Class Exercise 11

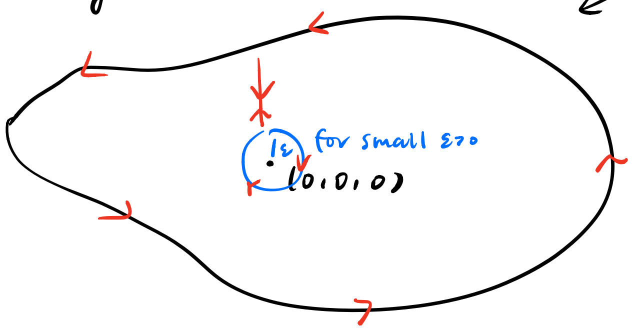
1. Let $f = (x^2 + y^2 + z^2)^{-1/2}$ and $\mathbf{F} = \nabla f$. Show that its circulation around any simple closed curve in the xy-plane is equal to zero as long as the curve does not touch the origin. You should distinguish two cases, that is, either the curve encloses the origin or not.

Case 1: Does not enclose the origin $\Rightarrow \vec{F}$ conservative

$\Rightarrow$ Standard result follows.

Case 2: Enclose the origin: Idea:

$\leftarrow$ see Tutorial notes.



2. Find the flux of the curl of the vector field $(x-y)\mathbf{i} + (y-z)\mathbf{j} + (z-x)\mathbf{k}$ across the surface S given by

$$\mathbf{r}(r,\theta) = r\cos\theta\mathbf{i} + r\sin\theta\mathbf{j} + (5-r)\mathbf{k}\ , \quad r \in [0,5],\ \theta \in [0, 2\pi].$$

$\partial_x := \frac{\partial}{\partial x}$

$\partial_y := \frac{\partial}{\partial y}$

$\partial_z := \frac{\partial}{\partial z}$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ x-y & y-z & z-x \end{vmatrix} = (1,1,1)$$

$$\frac{\partial \vec{r}}{\partial r} = (\cos\theta, \sin\theta\ -1)$$

$$\frac{\partial \vec{r}}{\partial \theta} = (-r\sin\theta, r\cos\theta, 0)$$

$$\frac{\partial \vec{r}}{\partial r} \times \frac{\partial \vec{r}}{\partial \theta} = \cdots = (r\cos\theta, r\sin\theta, r)$$

$$\vec{n} = \frac{\vec{r}_r \times \vec{r}_\theta}{|\vec{r}_r \times \vec{r}_\theta|} = \frac{(r\cos\theta, r\sin\theta, r)}{\sqrt{2}\, r} = \frac{1}{\sqrt{2}}(\cos\theta, \sin\theta, r)$$

$$\iint_S (\nabla \times \vec{F}) \cdot \vec{n}\, d\sigma = \int_0^{2\pi}\int_0^5 (1,1,1) \cdot \frac{1}{\sqrt{2}}(\cos\theta, \sin\theta, r) \cdot \sqrt{2}\, r\, dr\, d\theta$$

$$= \cdots$$