

## Chapter 2: Limits

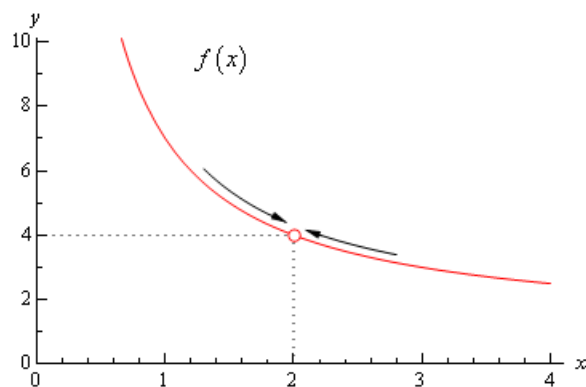
**Learning Objectives:**

- (1) Examine the limit concept and general properties of limits.
- (2) Compute limits using a variety of techniques.
- (3) Compute and use one-sided limits.
- (4) Investigate limits involving infinity and “ $e$ ”.

**2.1 Limit of a function at one point**

(Heuristic) “**Definition** 2.1.1. If  $f(x)$  gets “closer and closer” to a number  $L$  as  $x$  gets “closer and closer” to  $c$  from *both sides*, then  $L$  is called the **limit** of  $f(x)$  as  $x$  approaches  $c$ , denoted by

$$\lim_{x \rightarrow c} f(x) = L.$$



*Remark.* Limits are defined rigorously via “ $\varepsilon - \delta$ ” language.

**Example 2.1.1.** Let  $f(x) := x + 1$ . Find  $\lim_{x \rightarrow 1} f(x)$

|        |     |      |       |   |       |      |     |
|--------|-----|------|-------|---|-------|------|-----|
| $x$    | 0.9 | 0.99 | 0.999 | 1 | 1.001 | 1.01 | 1.1 |
| $f(x)$ | 1.9 | 1.99 | 1.999 | 2 | 2.001 | 2.01 | 2.1 |

When  $x$  approaches 1 from both sides,  $f(x)$  approaches 2. Therefore  $\lim_{x \rightarrow 1} f(x) = 2$ .

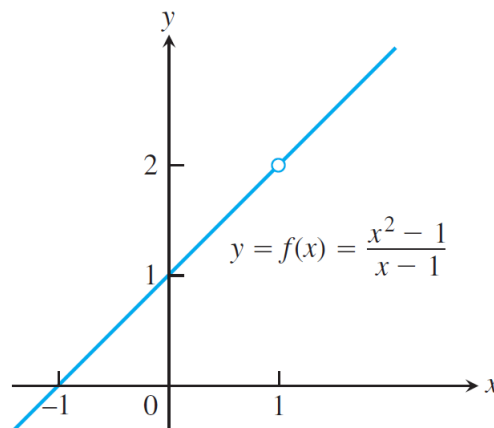
*Remark.* 1. The table only gives you an intuitive idea, this is **not** a rigorous proof.  
 2. **Don't** think that the limit is always obtained by substituting  $x = 1$  into  $f(x)$ . The limit only depends on the behavior of  $f(x)$  **near**  $x = 1$ , **but not at**  $x = 1$ .

**Example 2.1.2.**  $f(x) = \begin{cases} x + 1 & \text{if } x \neq 1, \\ \text{undefined} & \text{if } x = 1. \end{cases}$

|        |     |      |       |                  |       |      |     |
|--------|-----|------|-------|------------------|-------|------|-----|
| $x$    | 0.9 | 0.99 | 0.999 | <b>1</b>         | 1.001 | 1.01 | 1.1 |
| $f(x)$ | 1.9 | 1.99 | 1.999 | <b>undefined</b> | 2.001 | 2.01 | 2.1 |

When  $x$  approaches 1 from both sides,  $f(x)$  approaches 2. Therefore  $\lim_{x \rightarrow 1} f(x) = 2$ .

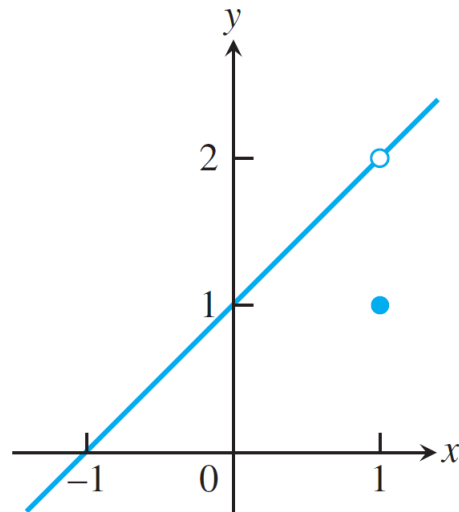
Disregard the value of  $f$  at 1, the limit of  $f(x)$  when  $x$  tends to 1 is always 2.



**Example 2.1.3.**  $f(x) = \begin{cases} x + 1 & \text{if } x \neq 1, \\ 1 & \text{if } x = 1. \end{cases}$

|        |     |      |       |          |       |      |     |
|--------|-----|------|-------|----------|-------|------|-----|
| $x$    | 0.9 | 0.99 | 0.999 | <b>1</b> | 1.001 | 1.01 | 1.1 |
| $f(x)$ | 1.9 | 1.99 | 1.999 | <b>1</b> | 2.001 | 2.01 | 2.1 |

When  $x$  approaches 1 from both sides,  $f(x)$  approaches 2. Therefore  $\lim_{x \rightarrow 1} f(x) = 2$ .

**Proposition 1.**

1. If  $f(x) = k$  is a constant function, then

$$\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} k = k.$$

For instance,  $\lim_{x \rightarrow 1} 9 = 9$ .

2. If  $f(x) = x$ , then

$$\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} x = c.$$

For instance,  $\lim_{x \rightarrow 3} x = 3$ .

**Proposition 2. (Algebraic properties of limits,  $+$ ,  $-$ ,  $\times$ ,  $\div$ )**

If  $\lim_{x \rightarrow c} f(x)$  and  $\lim_{x \rightarrow c} g(x)$  both exist (**important!**), then

$$1. \lim_{x \rightarrow c} (f(x) + g(x)) = \lim_{x \rightarrow c} f(x) + \lim_{x \rightarrow c} g(x)$$

$$2. \lim_{x \rightarrow c} (f(x) - g(x)) = \lim_{x \rightarrow c} f(x) - \lim_{x \rightarrow c} g(x)$$

$$3. \lim_{x \rightarrow c} (f(x)g(x)) = \lim_{x \rightarrow c} f(x) \cdot \lim_{x \rightarrow c} g(x)$$

Especially,  $\lim_{x \rightarrow c} k f(x) = k \lim_{x \rightarrow c} f(x)$  for any constant  $k$

$$4. \lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow c} f(x)}{\lim_{x \rightarrow c} g(x)} \quad \text{if } \lim_{x \rightarrow c} g(x) \neq 0.$$

$$5. \lim_{x \rightarrow c} (f(x))^p = \left[ \lim_{x \rightarrow c} f(x) \right]^p \quad \text{if } \left[ \lim_{x \rightarrow c} f(x) \right]^p \text{ exists}$$

**Example 2.1.4.** Compute the following limits:

1.  $\lim_{x \rightarrow 1} (x^3 + 2x - 5)$
2.  $\lim_{x \rightarrow 2} \frac{x^4 + x^2 - 1}{x^2 + 5}$
3.  $\lim_{x \rightarrow -2} \sqrt{4x^2 - 3}$

*Solution.*

1.  $\lim_{x \rightarrow 1} (x^3 + 2x - 5) = \lim_{x \rightarrow 1} x^3 + \lim_{x \rightarrow 1} 2x - \lim_{x \rightarrow 1} 5 = 1^3 + 2 \cdot 1 - 5 = -2.$
2.  $\lim_{x \rightarrow 2} \frac{x^4 + x^2 - 1}{x^2 + 5} = \frac{\lim_{x \rightarrow 2} (x^4 + x^2 - 1)}{\lim_{x \rightarrow 2} (x^2 + 5)} = \frac{\lim_{x \rightarrow 2} x^4 + \lim_{x \rightarrow 2} x^2 - \lim_{x \rightarrow 2} 1}{\lim_{x \rightarrow 2} x^2 + \lim_{x \rightarrow 2} 5} = \frac{19}{9}.$
3.  $\lim_{x \rightarrow -2} \sqrt{4x^2 - 3} = \sqrt{\lim_{x \rightarrow -2} (4x^2 - 3)} = \sqrt{\lim_{x \rightarrow -2} 4x^2 - \lim_{x \rightarrow -2} 3} = \sqrt{16 - 3} = \sqrt{13}.$

■

*Remark.* Generalizing the arguments for the first example above: the limit of any polynomial function  $P(x)$ ,

$$\lim_{x \rightarrow c} P(x) = P(c).$$

*Exercise 2.1.1.* Compute the following limits:

$$\lim_{x \rightarrow 1} \frac{1}{x-1}; \quad \lim_{x \rightarrow 1} \left( x^2 - \frac{3x}{x+5} \right)$$

**Example 2.1.5. (Cancelling a common factor)**

Find the limit:

$$\lim_{x \rightarrow 1} \frac{x^2 - 1}{x^2 - 3x + 2}.$$

*Solution.* We **can't** directly use property of division of limit because the denominator  $\lim_{x \rightarrow 1} (x^2 - 3x + 2) = 1^2 - 3 \times 1 + 2 = 0.$

$$\begin{aligned} \lim_{x \rightarrow 1} \frac{x^2 - 1}{x^2 - 3x + 2} &= \lim_{x \rightarrow 1} \frac{(x-1)(x+1)}{(x-1)(x-2)} \\ &= \lim_{x \rightarrow 1} \frac{\cancel{(x-1)}(x+1)}{\cancel{(x-1)}(x-2)} \\ &= \lim_{x \rightarrow 1} \frac{x+1}{x-2} \\ &= \frac{1+1}{1-2} = -2. \end{aligned}$$

**Example 2.1.6.** Compute

$$\lim_{x \rightarrow 1} \frac{x^3 - 5x + 4}{x^2 + 2x - 3}.$$

*Solution.* Write  $p(x) = x^3 - 5x + 4$  and  $q(x) = x^2 + 2x - 3$ . Because  $p(1) = q(1) = 0$ ,  $x - 1$  is a factor of  $p(x)$  and  $q(x)$ . We obtain

$$p(x) = (x - 1)(x^2 + x - 4) \text{ and } q(x) = (x - 1)(x + 3).$$

Then

$$\begin{aligned} \lim_{x \rightarrow 1} \frac{x^3 - 5x + 4}{x^2 + 2x - 3} &= \lim_{x \rightarrow 1} \frac{(x - 1)(x^2 + x - 4)}{(x - 1)(x + 3)} \\ &= \lim_{x \rightarrow 1} \frac{x^2 + x - 4}{x + 3} \\ &= \frac{1^2 + 1 - 4}{1 + 3} = -\frac{1}{2}. \end{aligned}$$

**Example 2.1.7. (Rationalization)**

Let  $f : [0, \infty) \setminus \{1\} \rightarrow \mathbf{R}$  defined by  $f(x) = \frac{\sqrt{x} - 1}{x - 1}$ . Find  $\lim_{x \rightarrow 1} f(x)$ .

*Solution.* For  $x \neq 1$ ,

$$\frac{\sqrt{x} - 1}{x - 1} = \frac{\sqrt{x} - 1}{x - 1} \cdot \frac{\sqrt{x} + 1}{\sqrt{x} + 1} = \frac{x - 1}{(x - 1)(\sqrt{x} + 1)} = \frac{1}{\sqrt{x} + 1}.$$

Hence

$$\lim_{x \rightarrow 1} \frac{\sqrt{x} - 1}{x - 1} = \lim_{x \rightarrow 1} \frac{1}{\sqrt{x} + 1} = \frac{1}{2}.$$

**Example 2.1.8. (Rationalization and Cancellation)**

Find

$$\lim_{x \rightarrow 1} \frac{\sqrt{x} - 1}{x^2 - 1}.$$

*Solution.*

$$\begin{aligned} \lim_{x \rightarrow 1} \frac{\sqrt{x} - 1}{x^2 - 1} &= \lim_{x \rightarrow 1} \frac{(\sqrt{x} - 1)(\sqrt{x} + 1)}{(x + 1)(x - 1)(\sqrt{x} + 1)} \\ &= \lim_{x \rightarrow 1} \frac{\cancel{x - 1}}{(x + 1)(\cancel{x - 1})(\sqrt{x} + 1)} \\ &= \lim_{x \rightarrow 1} \frac{1}{(x + 1)(\sqrt{x} + 1)} = \frac{1}{4}. \end{aligned}$$

**Challenge Question:** Let  $f : \mathbf{R} \setminus \{1\} \rightarrow \mathbf{R}$  defined by  $f(x) = \frac{\sqrt[3]{x} - 1}{x - 1}$ .

Find  $\lim_{x \rightarrow 1} f(x)$ .

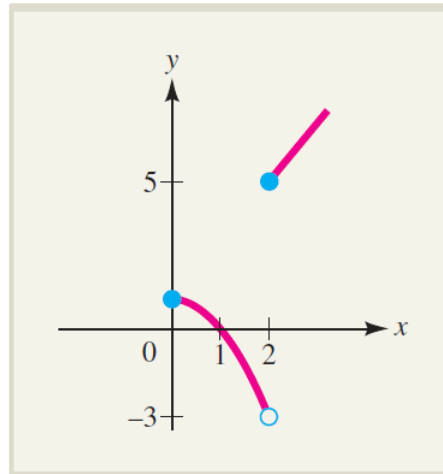
Hint:  $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$ .

**Proposition 3** (Composite functions/change of variables). If  $\lim_{x \rightarrow c} g(x) = k$  exists and  $\lim_{u \rightarrow k} f(u)$  exists, then  $\lim_{x \rightarrow c} f \circ g(x) = \lim_{u \rightarrow k} f(u)$ .

**Example 2.1.9.** Redo the last three examples using change of variables.

## 2.2 One-sided Limits

The following shows the graph of a piecewise function  $f(x)$ :



As  $x$  approaches 2 from the right,  $f(x)$  approaches 5 and we write

$$\lim_{x \rightarrow 2^+} f(x) = 5.$$

On the other hand, as  $x$  approaches 2 from the left,  $f(x)$  approaches -3 and we write

$$\lim_{x \rightarrow 2^-} f(x) = -3.$$

Limits of these forms are called **one-sided limits**. The limit is a **right-hand limit** if the approach is from the right. From the left, it is a **left-hand limit**.

**Definition 2.2.1.** If  $f(x)$  approaches  $L$  as  $x$  tends towards  $c$  from the left ( $x < c$ ), we write

$\lim_{x \rightarrow c^-} f(x) = L$ . It is called the **left-hand limit** of  $f(x)$  at  $c$ .

If  $f(x)$  approaches  $L$  as  $x$  tends towards  $c$  from the right ( $x > c$ ), we write  $\lim_{x \rightarrow c^+} f(x) = L$ .

It is called the **right-hand limit** of  $f(x)$  at  $c$ .

**Example 2.2.1.** Recall

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

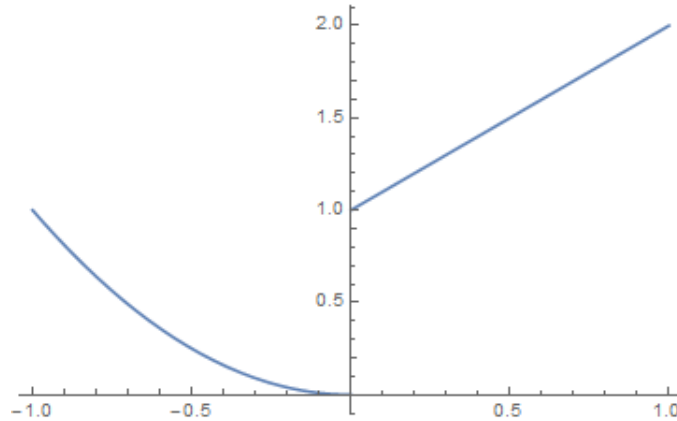
$$\lim_{x \rightarrow 0^+} |x| = \lim_{x \rightarrow 0^+} x = 0.$$

$$\lim_{x \rightarrow 0^-} |x| = \lim_{x \rightarrow 0^-} (-x) = 0.$$

For this case  $\lim_{x \rightarrow 0^+} |x| = \lim_{x \rightarrow 0^-} |x|$ . Then  $\lim_{x \rightarrow 0} |x| = 0$ .

**Example 2.2.2.** Define  $f : \mathbf{R} \rightarrow \mathbf{R}$ ,

$$f(x) = \begin{cases} x + 1 & \text{if } x \geq 0, \\ x^2 & \text{if } x < 0. \end{cases}$$



|        |           |           |           |   |       |      |     |
|--------|-----------|-----------|-----------|---|-------|------|-----|
| $x$    | -0.1      | -0.01     | -0.001    | 0 | 0.001 | 0.01 | 0.1 |
| $f(x)$ | $10^{-2}$ | $10^{-4}$ | $10^{-6}$ | 1 | 1.001 | 1.01 | 1.1 |

We have

$$\lim_{x \rightarrow 0^+} f(x) = 1.$$

and

$$\lim_{x \rightarrow 0^-} f(x) = 0.$$

*Remark.*

1. The left hand limit or the right hand limit may not be the same.
2. Does  $\lim_{x \rightarrow 0} f(x)$  exist? **No!**

**Proposition 4.**

$$\lim_{x \rightarrow c} f(x) = L \text{ if and only if } \lim_{x \rightarrow c^-} f(x) = L \text{ and } \lim_{x \rightarrow c^+} f(x) = L.$$

(i.e., both left hand limit and right hand limit exist and is equal to  $L$ )



**Example 2.2.3.** Suppose the function

$$f(x) = \begin{cases} x^2 + 1, & x \geq 1, \\ a, & x < 1. \end{cases}$$

has a limit as  $x$  approaches 1. Find the value of  $a$  and  $\lim_{x \rightarrow 1} f(x)$ .

*Solution.* Since  $\lim_{x \rightarrow 1} f(x)$  exists, we have

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1} f(x).$$

And

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} (x^2 + 1) = 2, \quad \lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} (a) = a.$$

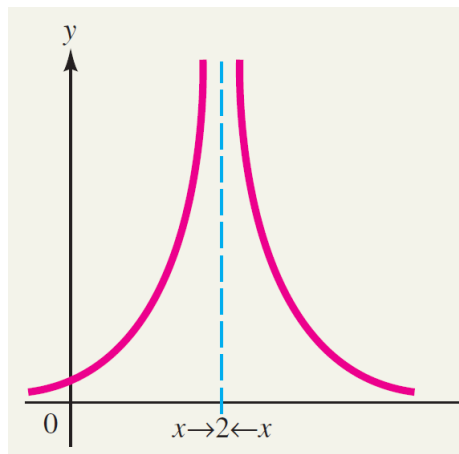
So,  $a = 2$ , and  $\lim_{x \rightarrow 1} f(x) = 2$ . ■

## 2.3 Infinite “Limits”

Consider the following limit

$$\lim_{x \rightarrow 2} \frac{1}{(x-2)^2}.$$

As  $x$  approaches 2, the denominator of the function  $f(x) = \frac{1}{(x-2)^2}$  approaches 0 and hence the value of  $f(x)$  becomes very large.



The function  $f(x)$  increases without bound as  $x \rightarrow 2$  both from left and from right. In this case, the limit *DNE* (does not exist) at  $x = 2$ , but we express the asymptotic behaviour

of  $f$  near 2 symbolically as

$$\lim_{x \rightarrow 2} \frac{1}{(x-2)^2} = +\infty.$$

*Remark.*  $+\infty$  is just a symbol, not a real number.

**Example 2.3.1.**

$$\lim_{x \rightarrow 0} \frac{-1}{x^2} = -\infty.$$

**Definition 2.3.1.** We say that  $\lim_{x \rightarrow c} f(x)$  is an infinite limit if  $f(x)$  increases or decreases without bound as  $x \rightarrow c$ .

If  $f(x)$  increases without bound as  $x \rightarrow c$ , we write

$$\lim_{x \rightarrow c} f(x) = +\infty.$$

If  $f(x)$  decreases without bound as  $x \rightarrow c$ , then

$$\lim_{x \rightarrow c} f(x) = -\infty.$$

**Example 2.3.2.** Evaluate

$$\lim_{x \rightarrow 2^+} \frac{x-3}{x^2-4} \text{ and } \lim_{x \rightarrow 2^-} \frac{x-3}{x^2-4}.$$

*Solution.*

$$\lim_{x \rightarrow 2^+} \frac{x-3}{x^2-4} = \lim_{x \rightarrow 2^+} \frac{x-3}{(x-2)(x+2)} = -\infty$$

since as  $x \rightarrow 2^+$ , we have  $x^2 - 4 = (x-2)(x+2) \rightarrow 0^+$  and  $x-3 \rightarrow -1^+$ .

$$\lim_{x \rightarrow 2^-} \frac{x-3}{x^2-4} = \lim_{x \rightarrow 2^-} \frac{x-3}{(x-2)(x+2)} = +\infty$$

since as  $x \rightarrow 2^-$ , we have  $x^2 - 4 = (x-2)(x+2) \rightarrow 0^-$  and  $x-3 \rightarrow -1^-$ .

■

**Exercise 2.3.1.** Find

$$\lim_{x \rightarrow \pi/2} \tan x; \quad \lim_{x \rightarrow \pi/2^-} \tan x; \quad \lim_{x \rightarrow \pi/2^+} \tan x; \quad \lim_{x \rightarrow 0^+} \ln x.$$

*Remark. Caveat!* When applying the rules in Proposition 2, roughly speaking:

- “ $a \pm \infty = \pm\infty$ ” when  $a$  is finite;
- “ $\infty + \infty = \infty$ ”; “ $-\infty - \infty = -\infty$ ”;
- “ $\infty \cdot \infty = \infty$ ”; “ $-\infty \cdot \infty = -\infty$ ”; “ $-\infty \cdot (-\infty) = \infty$ ”;
- “ $a \cdot \infty = \text{sign}(a)\infty$ ” when  $a \neq 0$ ;
- “ $\frac{a}{\pm\infty} = 0$ ” when  $a$  is finite;
- “ $\frac{a}{0^\pm} = \pm\text{sign}(a)\infty$ ” when  $a \neq 0$ ;
- but “ $\infty - \infty$ ”, “ $0 \cdot \infty$ ”, “ $\frac{\infty}{\infty}$ ”, “ $\frac{0}{0}$ ” can be quite arbitrary, and must be determined case by case! We shall introduce tools to compute limits of these forms later.

## 2.4 Limits at Infinity

**Definition 2.4.1.** If the values of the function  $f(x)$  approach the number  $L$  as  $x$  gets bigger and bigger (i.e. as  $x$  goes to  $+\infty$ ). Then  $L$  is called the limit of  $f(x)$  as  $x$  tends to  $+\infty$ . Denoted by

$$\lim_{x \rightarrow +\infty} f(x) = L.$$

Similarly we can define

$$\lim_{x \rightarrow -\infty} f(x) = M.$$

**Remark:** The value  $L$  and  $M$  may not be the same. If they are the same (i.e.,  $L = M$ ), we write

$$\lim_{x \rightarrow \infty} f(x) = L.$$

**Example 2.4.1.** Let  $f(x) = \frac{1}{x}$ .

|        |       |      |    |   |     |      |       |
|--------|-------|------|----|---|-----|------|-------|
| -1000  | -100  | -10  | -1 | 1 | 10  | 100  | 1000  |
| -0.001 | -0.01 | -0.1 | -1 | 1 | 0.1 | 0.01 | 0.001 |

$$\lim_{x \rightarrow \infty} \frac{1}{x} = \lim_{x \rightarrow +\infty} \frac{1}{x} = \lim_{x \rightarrow -\infty} \frac{1}{x} = 0.$$

**Proposition 5.** If  $A$  and  $k > 0$  are constants. Then

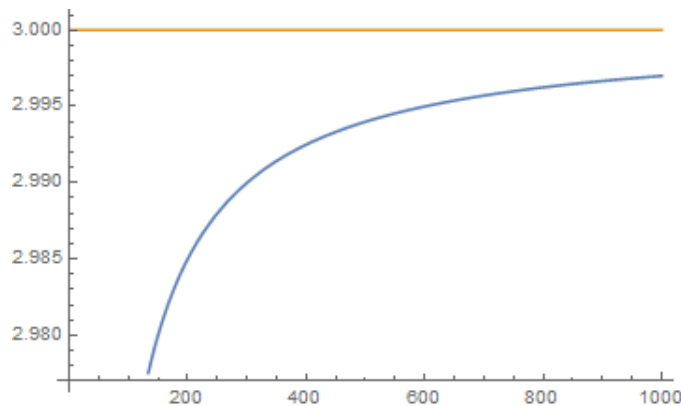
$$\lim_{x \rightarrow +\infty} \frac{A}{x^k} = 0 \quad \text{and} \quad \lim_{x \rightarrow -\infty} \frac{A}{x^k} = 0.$$

To determine the limit of a rational function as  $x \rightarrow \pm\infty$ , we can divide the numerator and denominator by the highest power of  $x$  in the **denominator**.

**Example 2.4.2.** Find  $\lim_{x \rightarrow +\infty} \frac{3x^2}{x^2 + x + 1}$

*Solution.*

$$\begin{aligned} & \lim_{x \rightarrow +\infty} \frac{3x^2}{x^2 + x + 1} \quad (\text{Divide both the top and bottom by } x^2) \\ &= \lim_{x \rightarrow +\infty} \frac{3}{1 + \frac{1}{x} + \frac{1}{x^2}} \\ &= \frac{3}{1 + 0 + 0} = 3. \end{aligned}$$



■

**Question:** Can we write

$$\lim_{x \rightarrow +\infty} \frac{3x^2}{x^2 + x + 1} = \frac{\lim_{x \rightarrow +\infty} (3x^2)}{\lim_{x \rightarrow +\infty} (x^2 + x + 1)}?$$

*Hint:* Recall the Caveat from the end of last section.

**Example 2.4.3.** Find  $\lim_{x \rightarrow +\infty} \frac{x - 1}{2x^2 + 3x + 1}$

*Solution.*

$$\begin{aligned} & \lim_{x \rightarrow +\infty} \frac{x-1}{2x^2+3x+1} \quad (\text{Divide both the top and bottom by } x^2) \\ &= \lim_{x \rightarrow +\infty} \frac{\frac{1}{x} - \frac{1}{x^2}}{2 + 3\frac{1}{x} + \frac{1}{x^2}} \\ &= \frac{0}{2+0+0} = 0. \end{aligned}$$

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**Example 2.4.4.** Find  $\lim_{x \rightarrow +\infty} \frac{x^3 - 1}{2x^2 + 3x + 1}$ .

*Solution.*

$$\begin{aligned} & \lim_{x \rightarrow +\infty} \frac{x^3 - 1}{2x^2 + 3x + 1} \\ &= \lim_{x \rightarrow +\infty} \frac{x - \frac{1}{x^2}}{2 + \frac{3}{x} + \frac{1}{x^2}} \\ &= +\infty. \end{aligned}$$

■

**Proposition 6.** Suppose

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0, a_n \neq 0$$

$$q(x) = b_m x^m + b_{m-1} x^{m-1} + \cdots + b_0, b_m \neq 0$$

Then

$$\lim_{x \rightarrow +\infty} \frac{p(x)}{q(x)} = \begin{cases} \frac{a_n}{b_m} & \text{if } n = m, \\ 0 & \text{if } n < m, \\ +\infty & \text{if } a_n b_m > 0, n > m, \\ -\infty & \text{if } a_n b_m < 0, n > m. \end{cases}$$

*Remark.* One way to see this: The leading term in a polynomial dominates the lower order terms as  $x \rightarrow \pm\infty$ . (Higher powers of  $x$  “grows faster” than lower powers of  $x$  as  $x \rightarrow \infty$ . Log functions grows slower than any polynomial function because (as we’ll see later)

$$\lim_{x \rightarrow \infty} \frac{\ln x}{x^a} = 0 \text{ for any } a > 0.$$

**Example 2.4.5.** Find  $\lim_{x \rightarrow \infty} \frac{3x^3 - 2x^2 + 1}{-x^3 + 7}$ .

*Solution.* By the proposition, the answer is  $\frac{3}{-1} = -3$ .

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Similar technique can be used for functions with radical (i.e., something like  $\sqrt{x}$ ).

**Example 2.4.6.** Find  $\lim_{x \rightarrow +\infty} \frac{3x - 1}{\sqrt{3x^2 + 1}}$ .

*Solution.* The term with highest degree of the denominator is  $x^2$ . But we need to take square root. So we divide the nominator and the denominator by  $\sqrt{x^2} = x$ . We have

$$\begin{aligned} \lim_{x \rightarrow +\infty} \frac{3x - 1}{\sqrt{3x^2 + 1}} &= \lim_{x \rightarrow +\infty} \frac{\frac{1}{x}(3x - 1)}{\frac{1}{x}\sqrt{3x^2 + 1}} \\ &= \lim_{x \rightarrow +\infty} \frac{3 - \frac{1}{x}}{\sqrt{3 + \frac{1}{x^2}}} = \frac{3}{\sqrt{3}} = \sqrt{3}. \end{aligned}$$

■

**Example 2.4.7. (Rationalization)**

Evaluate

$$\lim_{x \rightarrow +\infty} (\sqrt{x+1} - \sqrt{x}).$$

*Solution.* (Recall the *Caveat* from last section!)

$$\begin{aligned} \lim_{x \rightarrow +\infty} (\sqrt{x+1} - \sqrt{x}) &= \lim_{x \rightarrow +\infty} \frac{(\sqrt{x+1} - \sqrt{x})(\sqrt{x+1} + \sqrt{x})}{\sqrt{x+1} + \sqrt{x}} \\ &= \lim_{x \rightarrow +\infty} \frac{1}{\sqrt{x+1} + \sqrt{x}} \\ &= 0. \end{aligned}$$

■

*Exercise 2.4.1.*

$$1. \lim_{x \rightarrow -\infty} \frac{x^3 + 1}{-2x^3 + x} = -\frac{1}{2}.$$

$$2. \lim_{x \rightarrow -\infty} \frac{x}{\sqrt{x^2 + 1}} = -1 \quad (\text{Caution: } x < 0 \Rightarrow \frac{1}{x} = -\sqrt{\frac{1}{x^2}}).$$

$$3. \lim_{x \rightarrow +\infty} (\sqrt{x^2 + x} - \sqrt{x^2 - 2}) = \frac{1}{2}.$$

**Example 2.4.8.**  $\lim_{x \rightarrow +\infty} \sin x = ?$

## 2.5 Limits involving “e”

**Definition 2.5.1.**

$$e = \lim_{x \rightarrow +\infty} \left(1 + \frac{1}{x}\right)^x = \lim_{x \rightarrow -\infty} \left(1 + \frac{1}{x}\right)^x.$$

$e$  is the base for natural log,  $\log_e x = \ln x$ .

$$e = 2.71828\dots$$

| $x$                              | -1000   | -100    | -10     | 10      | 100     | 1000    |
|----------------------------------|---------|---------|---------|---------|---------|---------|
| $\left(1 + \frac{1}{x}\right)^x$ | 2.71964 | 2.73200 | 2.86797 | 2.59374 | 2.70481 | 2.71692 |

*Remark.* 1. Note that

$$e := \lim_{x \rightarrow +\infty} \left(1 + \frac{1}{x}\right)^x \neq \left(\lim_{x \rightarrow +\infty} 1 + \frac{1}{x}\right)^x = 1!$$

2. Motivation for defining  $e$  this way will be clear later when we learn about differentiation.

**Example 2.5.1.** Evaluate

$$\lim_{x \rightarrow +\infty} \left(1 - \frac{1}{x}\right)^x.$$

*Solution.*

$$\begin{aligned} \lim_{x \rightarrow +\infty} \left(1 - \frac{1}{x}\right)^x &= \lim_{x \rightarrow +\infty} \left[ \left(1 + \frac{1}{(-x)}\right)^{(-x)} \right]^{-1} && (\text{set } -x = y) \\ &= \left[ \lim_{y \rightarrow -\infty} \left(1 + \frac{1}{y}\right)^y \right]^{-1} \\ &= e^{-1} \end{aligned}$$

■

**Exercise 2.5.1.** Evaluate  $\lim_{x \rightarrow +\infty} \left(1 + \frac{2}{x}\right)^{2x} = e^4$ .