1. Compare Theorem (1), Theorem (2), Theorem (3):

Theorem (1).

Let B be a subset of $\mathbb{R}$. The statements below hold:

(a) *For any $x \in B$, $x \leq x$.*

(b) *For any $x, y \in B$, if $x \leq y$ and $y \leq x$ then $x = y$.*

(c) *For any $x, y, z \in B$, if $x \leq y$ and $y \leq z$ then $x \leq z$.*

Theorem (2).

The statements below hold:

(a) *Suppose $x \in \mathbb{N}$. Then x is divisible by x.*

(b) *Let $x, y \in \mathbb{N}$. Suppose y is divisible by x and x is divisible by y. Then $x = y$.*

(c) *Let $x, y, z \in \mathbb{N}$. Suppose y is divisible by x and z is divisible by y. Then z is divisible by x.*

Suppose we write $u | v$ exactly when v is divisible by u.
Then (a), (b), (c) become:

(a) For any $x \in \mathbb{N}$, $x | x$.
(b) For any $x, y \in \mathbb{N}$, if $x | y$ any $y | x$ then $x = y$.
(c) For any $x, y, z \in \mathbb{N}$, if $x | y$ and $y | z$ then $x | z$.

Theorem (1).
Let B be a subset of $\mathbb{R}$. The statements below hold:

(a) *For any $x \in B$, $x \leq x$.*
(b) *For any $x, y \in B$, if $x \leq y$ and $y \leq x$ then $x = y$.*
(c) *For any $x, y, z \in B$, if $x \leq y$ and $y \leq z$ then $x \leq z$.*

Theorem (3).
Let E be a set. The statements below hold:

(a) *For any $A \in \mathfrak{P}(E)$, $A \subset A$.*
(b) *For any $A, B \in \mathfrak{P}(E)$, if $A \subset B$ and $B \subset A$ then $A = B$.*
(c) *For any $A, B, C \in \mathfrak{P}(E)$, if $A \subset B$ and $B \subset C$ then $A \subset C$.*

Theorem (1), Theorem (2), Theorem (3) suggest the presence of some common structure for various mathematical objects. This mathematical structure is usually referred to as **partial ordering**.

2. **Definition.**

Let A be a set, and T be a relation in A with graph G.

(a) T *is said to be* **reflexive** *if the statement* (ρ) *holds:*

(ρ): *For any* $x \in A$, $(x, x) \in G$.

(b) T *is said to be* **anti-symmetric** *if the statement* (α) *holds:*

(α): *For any* $x, y \in A$, *if* $((x, y) \in G$ *and* $(y, x) \in G)$ *then* $x = y$.

(c) T *is said to be* **transitive** *if the statement* (τ) *holds:*

(τ): *For any* $x, y, z \in A$, *if* $((x, y) \in G$ *and* $(y, z) \in G)$ *then* $(x, z) \in G$.

Remark. The notions of reflexivity, anti-symmetry, and transitivity are 'logically independent' of each other.

3. **Definition.**

Let A be a set, and T be a relation in A with graph G.

T *is said to be a* **partially ordering** *in A if T is reflexive, anti-symmetric and transitive.*

We may also say that A is partially ordered by T. We may refer to the ordered pair (A, T) *as a* **poset**.

4. **Example (A). (Usual ordering for real numbers.)**

Theorem (1), which is concerned with the usual ordering for real numbers, can be re-formulated as:

Suppose B is a subset of $\mathbb{R}$. Define $G = \{(x, y) \mid x, y \in B \text{ and } x \leq y\}$. Then (B, B, G) is a partial ordering.

How?

By the definition of G, for any $x, y \in B$, $(x, y) \in G$ iff $x \leq y$.

So, in terms of G, the statement of Theorem (1) becomes:

Let B be a subset of $\mathbb{R}$. The statements below hold:

(a) For any $x \in B$, $\underbrace{(x, x) \in G}_{x \leq x}$.

(b) For any $x, y \in B$, if $\underbrace{(x, y) \in G}_{x \leq y}$ and $\underbrace{(y, x) \in G}_{y \leq x}$ then $x = y$.

(c) For any $x, y, z \in B$, if $\underbrace{(x, y) \in G}_{x \leq y}$ and $\underbrace{(y, z) \in G}_{y \leq z}$ then $\underbrace{(x, z) \in G}_{x \leq z}$.

Example (A). (Usual ordering for real numbers.)

Theorem (1), which is concerned with the usual ordering for real numbers, can be reformulated as:

Suppose B is a subset of $\mathbb{R}$. Define $G = \{(x, y) \mid x, y \in B$ and $x \leq y\}$.
Then (B, B, G) is a partial ordering.

Remark. Example (A) is the primordial example of partial orderings. The notations and terminologies for general partial orderings, soon to be introduced, are inspired by the usual ordering for real numbers.

We may think of the usual orderings in $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ as 'restrictions' to these sets of the usual ordering for real numbers.

5. **Lemma (4).**

Let A be a set. Suppose T is a partial ordering in A with graph G.
Then, for any subset C of A, $(C, C, G \cap C^2)$ is a partial ordering in C.

Remarks on terminologies and notations.

We call the partial ordering $(C, C, G \cap C^2)$ the **restriction** of T to C.

6. **Example (B). (Divisibility for natural numbers.)**

Theorem (2), which is concerned with divisibility for natural numbers, can be re-formulated as:

Define $G_{\text{div}} = \{(x, y) \mid x, y \in \mathbb{N}$ *and* y *is divisible by* $x\}$, *and* $T_{\text{div}} = (\mathbb{N}, \mathbb{N}, G_{\text{div}})$. *Then* T_{div} *is a partial ordering in* $\mathbb{N}$.

We call T_{div} the partial ordering in $\mathbb{N}$ defined by divisibility.

Remark. By Lemma (4), the restriction of T_{div} to any subset B of $\mathbb{N}$ defines a partial ordering in B.

Example. Suppose $B = [\![0, 9]\!]$. Then $(B, B, G_{div} \cap B^2)$ is a partial ordering, with

$$G_{div} \cap B^2 = \{ (0,0),$$
$$(1,0), (1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (1,7), (1,8), (1,9),$$
$$(2,0), (2,2), (2,4), (2,6), (2,8),$$
$$(3,0), (3,3), (3,6), (3,9),$$
$$(4,0), (4,4), (4,8),$$
$$(5,0), (5,5),$$
$$(6,0), (6,6),$$
$$(7,0), (7,7),$$
$$(8,0), (8,8),$$
$$(9,0), (9,9)\}.$$

Example (B). (Divisibility for natural numbers.)

Theorem (2), which is concerned with divisibility for natural numbers, can be re-formulated as:

> *Define* $G_{\text{div}} = \{(x, y) \mid x, y \in \mathbb{N}$ *and* y *is divisible by* $x\}$, *and* $T_{\text{div}} = (\mathbb{N}, \mathbb{N}, G_{\text{div}})$.
> *Then* T_{div} *is a partial ordering in* $\mathbb{N}$.

We call T_{div} the partial ordering in $\mathbb{N}$ defined by divisibility.

Remark. By Lemma (4), the restriction of T_{div} to any subset B of $\mathbb{N}$ defines a partial ordering in B.

Further remark. Although the usual ordering for natural numbers and T_{div} are both partial orderings in $\mathbb{N}$, there is a subtle but important difference between them:

- Every pair of natural numbers can be 'compared' in terms of the usual ordering. This is more formally formulated as:

  For any $x, y \in \mathbb{N}$, $x \leq y$ *or* $y \leq x$.

- Not every pair of natural numbers can be 'compared' in terms of T_{div}. This is more formally formulated as:

  There exists some $x, y \in \mathbb{N}$ *such that* $(x, y) \notin G_{\text{div}}$ *and* $(y, x) \notin G_{\text{div}}$.

  For instance, $(2, 3) \notin G_{\text{div}}$ and $(3, 2) \notin G_{\text{div}}$.

7. **Definition.**

Let A be a set, and T be a partial ordering in A with graph G.

(a) *Let $x, y \in A$. We say that x, y are* T**-comparable** *if $(x, y) \in G$ or $(y, x) \in G$.*

(b) We say that T is **strongly connected** (or **connex**) if the statement (κ) holds:

(κ): *For any $x, y \in A$, $(x, y) \in G$ or $(y, x) \in G$.*

8. **Definition.**

Let A be a set, and T be a partial ordering in A with graph G.

(a) *T is called a* **total ordering** *in A if T is strongly connected.*

We may also say that A is totally ordered by T, and that the poset (A, T) is totally ordered.

(b) *Let C be a subset of A.*

The set C is called a **chain** *with respect to T if the restriction of T to C is a total ordering in C.*

9. **Lemma (5).**

Suppose A be a set, and T is a partial ordering in A. Then the statements below are logically equivalent:

(a) *T is strongly connected.*

(b) *T is a total ordering in A.*

(c) *A is a chain with respect to T.*

Moreover, if T is a total ordering in A, then for any subset B of A, the restriction of T to B is a total ordering in B.

10. **Examples and non-examples on total orderings and chains.**

(a) Refer to Example (A).
For any subset B of $\mathbb{R}$, the usual ordering in B defines a total ordering in B.

(b) Refer to Example (B).
The partial ordering T_{div} in $\mathbb{N}$ defined by divisibility is not a total ordering in $\mathbb{N}$.

There are many proper subsets of $\mathbb{N}$ which are chains with respect to T_{div}, for instance:

$$\{0\}, \qquad \{2^k \mid k \in \mathbb{N}\}, \qquad \{3^k \mid k \in \mathbb{N}\}.$$

However, none of the sets below is a chain with respect to T_{div}:

$$\underbrace{\{2^j \mid j \in \mathbb{N}\} \cup \{3^k \mid k \in \mathbb{N}\}}_{\substack{\text{2, 3 belong to this set} \\ \text{but they are not} \\ T_{\text{div}}\text{-comparable.}}}, \qquad \underbrace{\{2^j \cdot 3^k \mid j, k \in \mathbb{N}\}}_{\substack{\text{2, 3 belong to this set} \\ \text{but they are not} \\ T_{\text{div}}\text{-comparable.}}}.$$

11. **Example (C). (Subset relation.)**

Theorem (3), which is concerned with the subset relation within an arbitrarily given set, can be re-formulated as:

> *Suppose E is a set. Define $G_{E,\text{subset}} = \{(U, V) \mid U, V \in \mathfrak{P}(E) \text{ and } U \subset V\}$, and $T_{E,\text{subset}} = (\mathfrak{P}(E), \mathfrak{P}(M), G_{E,\text{subset}})$.*
> *Then $T_{E,\text{subset}}$ is a partial ordering in $\mathfrak{P}(E)$.*

We call $T_{E,\text{subset}}$ the partial ordering in $\mathfrak{P}(E)$ defined by the subset relation.

When E contains two or more elements, $T_{E,\text{subset}}$ is not a total ordering.

By Lemma (4), the restriction of the partial ordering $T_{E,\text{subset}}$ to any subset of $\mathfrak{P}(E)$ defines a partial ordering on that subset of $\mathfrak{P}(E)$.

Example. Suppose $E = \{0, 1, 2\}$. Then $\mathfrak{P}(E) = \{\emptyset, \{0\}, \{1\}, \{2\}, \{1,2\}, \{0,2\}, \{0,1\}, E\}$.
Write $a = \{0\}$, $b = \{1\}$, $c = \{2\}$, $a' = \{1, 2\}$, $b' = \{0, 2\}$, $c' = \{0, 1\}$.
The graph $G_{E,\text{subset}}$ of the partial ordering $T_{E,\text{subset}}$ is given by

$$\begin{aligned} G_{E,\text{subset}} = \{ & (\emptyset, \emptyset), (\emptyset, a), (\emptyset, b), (\emptyset, c), (\emptyset, a'), (\emptyset, b'), (\emptyset, c'), (\emptyset, E), \\ & (a, a), (a, b'), (a, c'), (a, E), (b, b), (b, a'), (b, c'), (b, E), \\ & (c, c), (c, a'), (c, b'), (c, E), (a', a'), (a', E), (b', b'), (b', E), (c', c'), (c', E), \\ & (E, E) \}. \end{aligned}$$

12. **Conventions on notations for partial orderings.**

We are going to introduce some conventions on notations for general partial orderings. They are inspired by the notations for usual orderings for real numbers and those for subset relations.

Let A be a set and T be a partial ordering in A with graph G.

Suppose we agree to write $(x, y) \in G$ as $x \preceq_T y$.

We pronounce '$x \preceq_T y$' as

'*x precedes or equals y under the partial ordering T*'.

(a) If T is the only partial ordering in A under consideration, we may drop the reference to the symbols T, G and write:

- '$x \preceq y$' in place of ' $x \preceq_T y$;
- '*A is partially ordered by $\preceq$*' in place of *A is partially ordered by T*;
- '*$(A, \preceq)$ is a poset*' in place of *(A, T) is a poset*; et cetera.

Under the above conventions, the statements (ρ), (α), (τ) that hold for the partial ordering T are re-formulated as:

(ρ): *For any $x \in A$, $x \preceq x$.*

(α): *For any $x, y \in A$, if ($x \preceq y$ and $y \preceq x$) then $x = y$.*

(τ): *For any $x, y, z \in A$, if ($x \preceq y$ and $y \preceq z$) then $x \preceq z$.*

(b) We also agree that the same symbol $\preceq$ will be used for the restriction of T to any subset of A.

(c) We may write '$x \preceq_T y$' as '$y \succeq_T x$'. The latter is pronounced as

'*y succeeds or equals x under the partial ordering T*'

(d) We may write $x \prec_T y$, or equivalently, $y \succ_T x$, exactly when $(x, y) \in G$ and $x \neq y$.

We pronounce '$x \prec_T y$' as

'*x precedes y under the partial ordering T*'.

We pronounce '$y \succ_T x$' as

'*y succeeds x under the partial ordering T*'.

Warning. Care must be taken because of the visual resemblance between the symbol $\leq$ and the symbol $\preceq$.

When we are using the symbol '$\preceq$' for formulating statements concerned with a general partial ordering T in an arbitrary set A, we have to deliberately remind ourselves that the statement (κ) may fail to hold:

(κ): *For any $x, y \in A$, $x \preceq y$ or $y \preceq x$.*

In fact (κ) holds exactly when the partial ordering T is a total ordering in A.

13. **Lemma (6).**

Let A be a set, and T be a partial ordering in A with graph G. Write $u \preceq v$ exactly when $(u, v) \in G$.

(a) *Let $x, y \in A$. The statements below are logically equivalent:*

i. x, y are T-comparable. ($x \preceq y$ or $y \preceq x$.)

ii. *Exactly one of '$x \prec y$', '$x = y$', '$x \succ y$' is true.*

(b) *T is strongly connected iff the statement $(\tau\chi)$ holds:*

$(\tau\chi)$: *For any $x, y \in A$, exactly one of '$x \prec y$', '$x = y$', '$x \succ y$' is true.*

Remark. When T is indeed a total ordering in A, the statement $(\tau\chi)$ is known as the **Law of Trichotomy** in the poset (A, T).

Proof of (a).

Let $x, y \in A$.

- [(ii) $\Rightarrow$ (i) ?]

Suppose exactly one of '$x \prec y$', '$x = y$', '$x \succ y$' holds. Then by definition of $\preceq$, we have $x \preceq y$ or $y \preceq x$.

- [(i) $\Rightarrow$ (ii) ?]

Suppose $x \preceq y$ or $y \preceq x$. Note that $x = y$ or $x \neq y$.

(Case 1). Suppose $x = y$. Then '$x \prec y$' is false. So is '$x \succ y$'.

(Case 2). Suppose $x \neq y$. Then $x \prec y$ or $y \prec x$.

(Case 2a). Suppose $x \prec y$. Then '$y \prec x$' is false; otherwise by anti-symmetry $x = y$.

(Case 2b). Suppose $y \prec x$. Then '$x \prec y$' is false; otherwise by anti-symmetry $x = y$.

14. **Example (D). (Lexicographical ordering in $\mathbb{N}^2$.)**

With the usual ordering in $\mathbb{N}$, we are going to construct a total ordering in $\mathbb{N}^2$, which is inspired by how words in a dictionary are arranged according to alphabetical order.

Define $J = \left\{ ((s,t),(u,v)) \;\middle|\; \begin{array}{l} s,t,u,v \in \mathbb{N}, \text{ and} \\ [s < u \text{ or } (s = u \text{ and } t \leq v)] \end{array} \right\}$, and $T = (\mathbb{N}^2, \mathbb{N}^2, J)$. Note that $J \subset (\mathbb{N}^2)^2$.

With a straightforward calculation, we can verify that T is a total ordering in $\mathbb{N}^2$.

The total ordering T is called the **lexicographical ordering** in $\mathbb{N}^2$.

For any $s,t,u,v \in \mathbb{N}$, we write $(s,t) \leq_{\text{lex}} (u,v)$ exactly when $((s,t),(u,v)) \in J$.
Then by definition, $(s,t) \leq_{\text{lex}} (u,v)$ iff $[s < u \text{ or } (s = u \text{ and } t \leq v)]$.

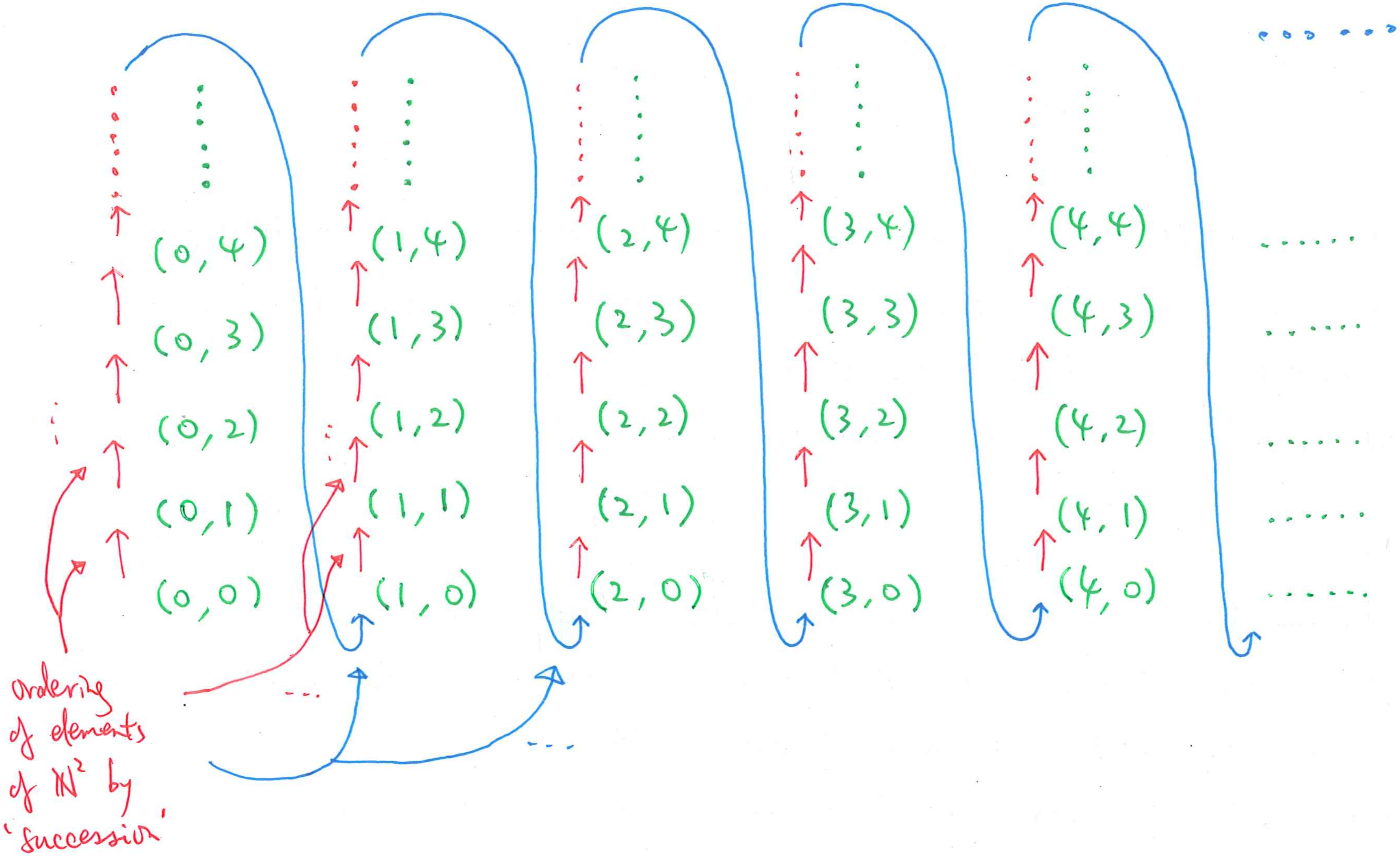
Illustrations:

- $(1,3) <_{\text{lex}} (2,0)$. (Reason: $1 < 2$.)
- $(2,3) <_{\text{lex}} (2,4)$. (Reason: $2 = 2$ and $3 < 4$.)

As a whole, T can be visualized as:

$$(0,0) \leq_{\text{lex}} (0,1) \leq_{\text{lex}} (0,2) \leq_{\text{lex}} \cdots \leq_{\text{lex}} (1,0) \leq_{\text{lex}} (1,1) \leq_{\text{lex}} \cdots \leq_{\text{lex}} (2,0) \leq_{\text{lex}} \cdots \leq_{\text{lex}} (3,0) \leq_{\text{lex}} \cdots \leq_{\text{lex}} (4,0) \leq_{\text{lex}} \cdots$$

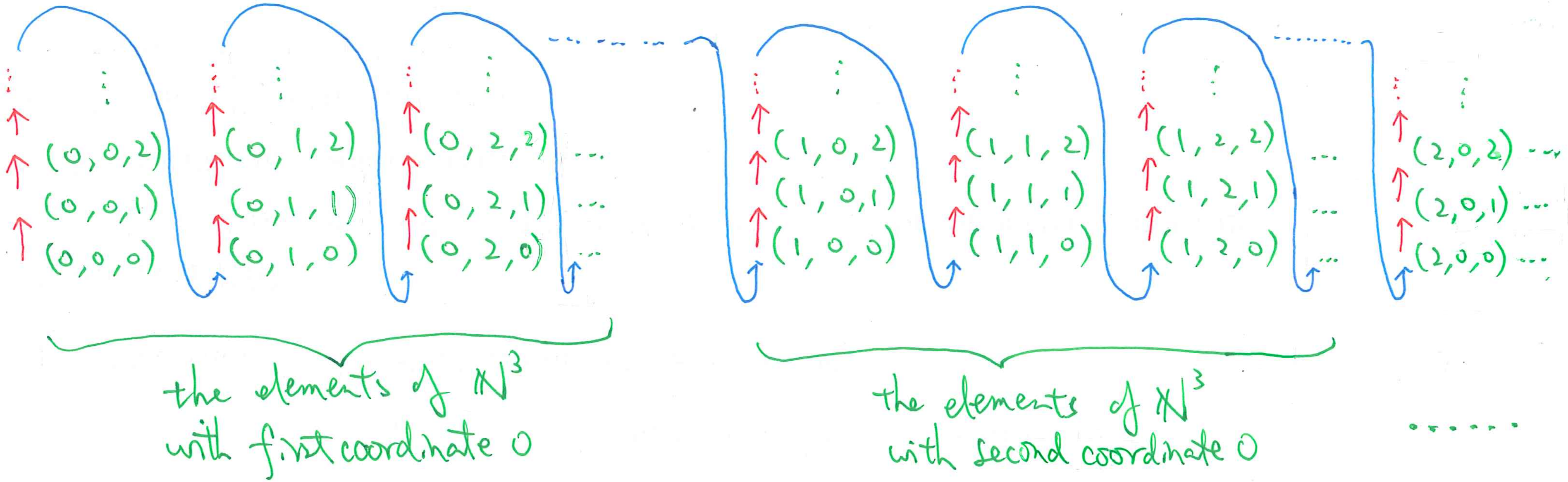
Visualization of the lexicographical ordering in $\mathbb{N}^2$.



Visualization of the lexicographical ordering in $\mathbb{N}^3$.

The graph J' of such a total ordering is given by

$$J' = \{((r,s,t),(u,v,w)) \mid r,s,t,u,v,w \in \mathbb{N}, \text{ and } [r<u \text{ or } (r=u \text{ and } s<v) \text{ or } (r=u \text{ and } s=v \text{ and } t \leq w)]\}.$$



Remark. We can apply the same method to construct the lexicographical ordering $\leq_{\text{lex}}$ in $\mathbb{N}^3$, given by $(\star_3)$:

$(\star)$ For any $r, s, t, u, v, w \in \mathbb{N}$, $(r, s, t) \leq_{\text{lex}} (u, v, w)$ iff $[r < u$ or $(r = u$ and $s < v)$ or $(r = u$ and $s = v$ and $t \leq w)]$.

We can 'inductively' construct the lexicographical ordering in $\mathbb{N}^k$ for each $k \in \mathbb{N}\backslash\{0\}$.

Further remark. The above constructions ultimately rely on the fact that the usual ordering in $\mathbb{N}$ is a total ordering in $\mathbb{N}$. No other aspect of the natural number system has anything to do with this construction.

Imitating the above construction, We can construct the lexicographical ordering $\leq_{\text{lex}}$ in $\mathbb{R}^2$ from the usual ordering in $\mathbb{R}$, which is given by $(\star)$:

$(\star)$ For any $s, t, u, v \in \mathbb{R}$, $(s, t) \leq_{\text{lex}} (u, v)$ iff $[s < u$ or $(s = u$ and $t \leq v)]$.

The lexicographical ordering in $\mathbb{N}^2$ is the restriction of this total ordering in $\mathbb{R}^2$.

We can 'inductively' construct the lexicographical ordering in $\mathbb{R}^k$ for each $k \in \mathbb{N}\backslash\{0\}$.

Example (D) is an illustration of the idea in Theorem (7), which is concerned with general partial orderings.

15. **Theorem (7).**

Let A, B be sets. Suppose R is a partial ordering in A with graph G, and S is a partial ordering in B with graph H.

Write $s \preceq_R u$ exactly when $(s, u) \in G$. Write $t \preceq_S v$ exactly when $(t, v) \in H$.

Define $J = \left\{ ((s,t),(u,v)) \;\middle|\; \begin{array}{l} s, u \in A, \text{ and } \; t, v \in B, \text{ and} \\ [s \prec_R u \text{ or } \; (s = u \text{ and } \; t \preceq_S v)] \end{array} \right\}$, *and* $T = (A \times B, A \times B, J)$.

Then T is a partial ordering in $A \times B$.

Moreover, if R is a total ordering in A and S is a total ordering in B, then T is a total ordering in $A \times B$.

Remark on terminologies and notations. T is called the **lexicographical ordering** in $A \times B$ induced by R and S.

16. **Definition.**

Let A be a set, and T be a partial ordering in A with graph G. Write $u \preceq v$ exactly when $(u, v) \in G$.

Let B be a subset of A.

Let $\lambda \in B$. We say λ is a $\left\{ \begin{array}{c} \textbf{greatest} \\ \textbf{least} \end{array} \right\}$ **element** *of B with respect to T if, for any $x \in B$,*

$$\left\{ \begin{array}{c} x \preceq \lambda \\ x \succeq \lambda \end{array} \right\}.$$

Remark. A subset of A has at most one greatest/least element with respect to T. Hence it makes sense to refer to such an element of A as 'the' greatest/least element with respect to T, if it exists.

Here in this Handout we focus on the question of existence of greatest/least elements for sets with respect to total orderings.

17. **Example (A'). (Usual ordering for real numbers.)**

The notion of greatest/least element for subsets of $\mathbb{R}$ with respect to the usual ordering for real numbers reduces to that for 'greatest/least element for subsets of $\mathbb{R}$', introduced in the Handout *Greatest/least element, upper/lower bound.*

(a) According to the Well-ordering Principle for Integers, for any subset B of $\mathbb{N}$, if B is non-empty, then B has a least element (with respect to the usual ordering for natural numbers).

A non-empty subset of $\mathbb{N}$ does not necessarily have any greatest element.

(b) Let a, b be real numbers. Supposed $a < b$.

	least element	greatest element
(a, b)	*nil*	*nil*
$[a, b)$	a	*nil*
$(a, b]$	*nil*	b
$[a, b]$	a	b

	least element	greatest element
$(a, +\infty)$	*nil*	*nil*
$[a, +\infty)$	a	*nil*
$(-\infty, b)$	*nil*	*nil*
$(-\infty, b]$	*nil*	b

18. **Definition.**

Let A be a set, and T be a partial ordering in A. We say T is a **well-order relation** *in A if the statement (λ) holds:*

(λ) *For any subset B of A, if B is non-empty then B has a least element with respect to T.*

We also say that A is **well-ordered** *by T, and that the poset (A, T) is well-ordered.*

Simple examples and non-examples of well-ordered sets.

(a) $\mathbb{N}$ is well-ordered by the usual ordering for natural numbers, according to Example (A').
(This is just a re-formulation of the statement of the Well-ordering Principle for Integers.)
This is the primordial example of well-ordered sets.

(b) Every non-empty subset of $\mathbb{Z}$ which is bounded below in $\mathbb{Z}$ is well-ordered by the usual ordering for integers.
$\mathbb{Z}$ is not well-ordered by the usual ordering for integers. (Why?)

(c) $\mathbb{Q}$ is not well-ordered by the usual ordering for rational numbers. (Why?)

(d) $\mathbb{R}$ is not well-ordered by the usual ordering for real numbers. (Why?)

19. **Lemma (8).**

Let A be a set, and T is a partial ordering in A.

Suppose A is well-ordered by T. Then A is totally ordered by T.

Proof of Lemma (8).

Let A be a set, and T be a partial ordering in A with graph G. Suppose A is well-ordered by T.

Pick any $x, y \in A$. Define $B = \{x, y\}$.

Then B is a non-empty subset of A.

By assumption, A is well-ordered by T.

Then B has a least element with respect to T, say, x.

Therefore, by definition, $(x, y) \in G$. Therefore $(x, y) \in G$ or $(y, x) \in G$.

It follows that A is totally ordered by T.

Non-examples on well-order relations.

According to Lemma (8), there is no chance for a partial ordering which is not a total ordering to be a well-order relation.

- Refer to Example (B).

  The partial ordering T_{div} in $\mathbb{N}$ defined by divisibility is not a well-order relation, because it is not a total ordering in $\mathbb{N}$.

- Refer to Example (C).

  When E is a set which has at least two elements, $(\mathfrak{P}(E), \mathfrak{P}(E), G_{E,\text{subset}})$ is not a well-order relation, because it is not a total ordering in $\mathfrak{P}(E)$.

Reminder. The converse of Lemma (8) is false: a total ordering in a set is not necessarily a well-order relation in that set.

(For instance, the usual ordering for real numbers is a total ordering in $\mathbb{R}$ but it is not a well-order relation in $\mathbb{R}$.)

20. **Lemma (9).**

Let A be a set. Suppose T is a well-order relation in A with graph G.

Then, for any subset B of A, $(B, B, G \cap B^2)$ is a well-order relation in B.

21. **Theorem (10).**

Let A be a non-empty set. Suppose T is a well-order relation in A with graph G. Write $x \preceq y$ iff $(x, y) \in G$.

Then the statements below hold:

(a) *There exists some unique $\lambda \in A$ such that for any $x \in A \backslash \{\lambda\}$, $\lambda \prec x$.*

(b) *For any $x \in A$, if x is not a greatest element of A with respect to T then there exists some unique $y \in A$ such that $x \prec y$ and (for any $z \in A$, if $x \preceq z \preceq y$ then $z = x$ or $z = y$).*

Remark. Theorem (10) brings out what is special about well-ordered posets.

- Statement (a) says that some unique element of A, namely the least element of A with respect to T, will be the 'starting element' of A, in the sense that no element of A will precede it with respect to T.
- Statement (b) says that it makes sense to talk about the (unique) 'next element' of A for each element of A, in the sense that no third element of A will be between these two.

This allows us to visualize the 'ordering' of all the elements of A, with respect to T, in the 'chain of inequalities'

$$\lambda \preceq \lambda' \preceq \lambda'' \preceq \lambda''' \preceq \cdots$$

in which

λ is the least element of A with respect to T,

λ' is the least element of $A \backslash \{\lambda\}$ with respect to T,

λ'' is the least element of $A \backslash \{\lambda, \lambda'\}$ with respect to T,

λ''' is the least element of $A \backslash \{\lambda, \lambda', \lambda''\}$ with respect to T, et cetera.

An illustration is how we may visualize the 'ordering' for all natural numbers with respect to its usual ordering:

$$0 \leq 1 \leq 2 \leq 3 \leq 4 \leq \cdots$$

This cannot be done for the usual ordering for integers because $\mathbb{Z}$ has no least element.

This cannot be done for the usual ordering for rational numbers because the notion of 'next rational' number does not make sense: between any two distinct rational numbers there is definitely a third rational number.

But we may ask:

- *Is it possible to equip these sets with some other partial orderings which are well-order relations?*

22. **Example (D'). (Lexicographical ordering in $\mathbb{N}^2$ as a well-order relation in $\mathbb{N}^2$.)**

The lexicographical ordering in $\mathbb{N}^2$ is a well-order relation in $\mathbb{N}^2$ because the statement (†) holds:

(†) *For any subset B of $\mathbb{N}^2$, if B is non-empty, then B has a least element with respect to the lexicographical ordering in $\mathbb{N}^2$.*

Below is the idea for the argument for the statement (†). (The detail is left as an exercise.)

Suppose B is a non-empty subset of $\mathbb{N}^2$.

Then we may pick some element of B, say, the ordered pair of natural numbers, say, (u, v).

The lexicographical ordering in $\mathbb{N}^2$ allows us to visualize the 'ordering' for all the elements of $\mathbb{N}^2$, up to and including (u, v), through such a 'chain of inequalities' below:

$$(0,0) \leq_{\text{lex}} (0,1) \leq_{\text{lex}} (0,2) \leq_{\text{lex}} \cdots \leq_{\text{lex}} (1,0) \leq_{\text{lex}} (1,1) \leq_{\text{lex}} \cdots \leq_{\text{lex}} (2,0) \leq_{\text{lex}} \cdots \leq_{\text{lex}} (u,0) \leq_{\text{lex}} (u,1) \leq_{\text{lex}} \cdots \leq_{\text{lex}} (u,v)$$

So elements of B are listed in at least one of the rows $(\sharp_0)$, $(\sharp_1)$, $(\sharp_2)$, ..., $(\sharp_u)$, each with 'constant' first coordinate, in the table below:

$$\begin{array}{lllllllllll}
(\sharp_0): & (0,0) & (0,1) & (0,2) & (0,3) & \cdots & (0,v-1) & (0,v) & (0,v+1) & \cdots \\
(\sharp_1): & (1,0) & (1,1) & (1,2) & (1,3) & \cdots & (1,v-1) & (1,v) & (1,v+1) & \cdots \\
(\sharp_2): & (2,0) & (2,1) & (2,2) & (2,3) & \cdots & (2,v-1) & (2,v) & (2,v+1) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \\
(\sharp_u): & (u,0) & (u,1) & (u,2) & (u,3) & \cdots & (u,v-1) & (u,v) & (u,v+1) & \cdots
\end{array}$$

an element of B.

The Well-ordering Principle for Integers guarantees that there will be a row in this table with the 'smallest value of label', say, s, so that some element of B, say, (s,w) is listed in the row $(\sharp_s)$.

There is no element of B within the rows $(\#_0), (\#_1), \dots, (\#_{s-1})$.

$$\begin{array}{lllllllllll}
(\sharp_0): & (0,0) & (0,1) & (0,2) & (0,3) & \cdots & (0,w-1) & (0,w) & (0,w+1) & \cdots \\
(\sharp_1): & (1,0) & (1,1) & (1,2) & (1,3) & \cdots & (1,w-1) & (1,w) & (1,w+1) & \cdots \\
(\sharp_2): & (2,0) & (2,1) & (2,2) & (2,3) & \cdots & (2,w-1) & (2,w) & (2,w+1) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \\
(\sharp_s): & (s,0) & (s,1) & (s,2) & (s,3) & \cdots & (s,w-1) & (s,w) & (s,w+1) & \cdots
\end{array}$$

$s \leq u$

an element of B.

Then the Well-ordering Principle for Integers further guarantees that amongst

$$(s,0),(s,1),(s,2),\cdots,(s,w-1),(s,w),$$

there will be an element of B with the 'smallest second coordinate', say, t.

There is no element of B within the rows $(\sharp_0), (\sharp_1), \cdots, (\sharp_{s-1})$.

$$\begin{array}{lllllllll}
(\sharp_0): & (0,0) & (0,1) & (0,2) & (0,3) & \cdots & (0,t-1) & (0,t) & (0,t+1) & \cdots \\
(\sharp_1): & (1,0) & (1,1) & (1,2) & (1,3) & \cdots & (1,t-1) & (1,t) & (1,t+1) & \cdots \\
(\sharp_2): & (2,0) & (2,1) & (2,2) & (2,3) & \cdots & (2,t-1) & (2,t) & (2,t+1) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \\
(\sharp_s): & (s,0) & (s,1) & (s,2) & (s,3) & \cdots & (s,t-1) & (s,t) & (s,t+1) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots &
\end{array}$$

There is no element of B within $(s,0), \ldots, (s,t-1)$.

an element of B

(s,t) will be the least element of B with respect to the lexicographical ordering in $\mathbb{N}^2$.

Example (D') is an illustration of the idea in Theorem (11).

23. **Theorem (11).**

Let A, B be sets. Suppose R is a well-order relation in A, and S is a well-order relation in B.

Then the lexicographical ordering in $A \times B$ induced by R and S is a well-order relation in $A \times B$.

24. **Example (E). (Well-order relation in $\mathbb{Z}$ arising from the usual ordering for natural numbers.)**

Recall that that $\mathbb{Z}$ is not well-ordered by the usual ordering for integers.

How, we may define a well-order relation in $\mathbb{Z}$ with the help of the usual ordering for natural numbers.

Define the function $f : \mathbb{Z} \longrightarrow \mathbb{N}$ by $f(x) = \begin{cases} 2x & \text{if } x \text{ is non-negative} \\ -2x - 1 & \text{if } x \text{ is negative} \end{cases}$.

f is an injective function from $\mathbb{Z}$ to $\mathbb{N}$.

Let $G = \{(x, y) \mid x \in \mathbb{Z} \text{ and } y \in \mathbb{Z} \text{ and } f(x) \leq f(y)\}$, and $S = (\mathbb{Z}, \mathbb{Z}, G)$.

S is a well-order relation in $\mathbb{Z}$.

So we visualize the 'ordering' for all integers with respect to the well-order relation S, through the 'chain of inequalities' below:

$$0 \preceq_S -1 \preceq_S 1 \preceq_S -2 \preceq_S 2 \preceq_S -3 \preceq_S 3 \preceq_S \cdots \preceq_S n-1 \preceq_S -n \preceq_S n \preceq_S \cdots$$

This is simply a direct translation, via f and S, of the chain of inequalities

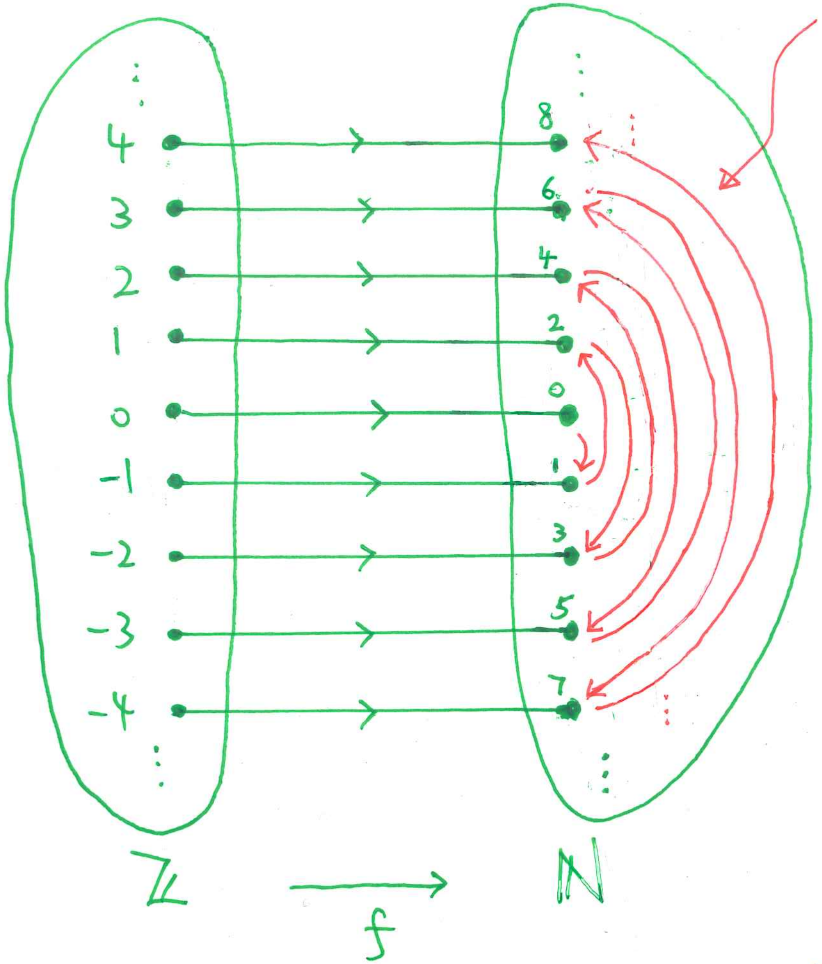
$$0 \leq 1 \leq 2 \leq 3 \leq 4 \leq 5 \leq 6 \leq \cdots \leq 2n-2 \leq 2n-1 \leq 2n \leq \cdots .$$

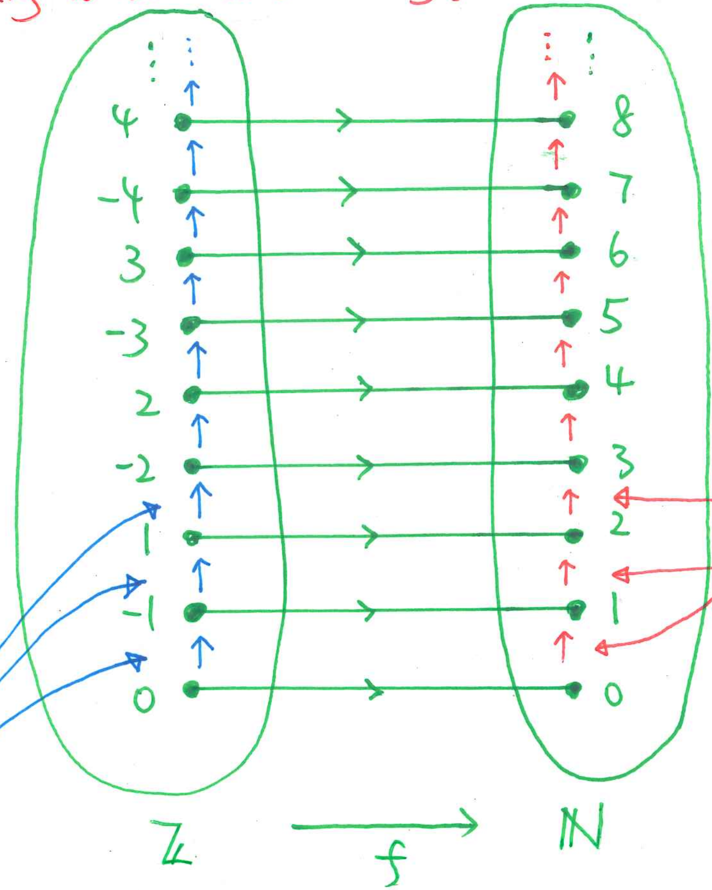
Note that $0 \prec_S -1 \prec_S 1$ whereas $-1 < 0 < 1$. Hence S is certainly distinct from the usual ordering for integers.

Visualization for Example (E).

The injective function $f: \mathbb{Z} \to \mathbb{N}$ is given by $f(x) = \begin{cases} 2x & \text{if } x \text{ is non-negative.} \\ -2x-1 & \text{if } x \text{ is negative.} \end{cases}$

This gives the ordering of natural numbers by succession according to the usual ordering for natural numbers.





This is the partial ordering S in $\mathbb{Z}$ defined by the pullback of the usual ordering for natural numbers by the injective function f. It is a well-order relation in $\mathbb{Z}$ because the usual ordering for natural numbers is a well-order relation in $\mathbb{N}$.

25. **Example (F). (Well-order relation in $\mathbb{N}^2$ which is not the same as the lexicographical ordering.)**

Recall that the lexicographical ordering in $\mathbb{N}^2$ is a well-order relation in $\mathbb{N}^2$. We now introduce, via an injective function from $\mathbb{N}^2$ to $\mathbb{N}$, another well-order relation in $\mathbb{N}^2$ which is not the lexicographical ordering in $\mathbb{N}^2$.

Define the function $f : \mathbb{N}^2 \longrightarrow \mathbb{N}$ by $f(x, y) = 2^x 3^y$ for any $x, y \in \mathbb{N}$.

f is an injective function from $\mathbb{N}^2$ to $\mathbb{N}$. (You need Euclid's Lemma to justify this claim.)

Let $G = \{((s, t), (u, v)) \mid s, t, u, v \in \mathbb{N} \text{ and } f(s, t) \leq f(u, v)\}$, and $S = (\mathbb{N}^2, \mathbb{N}^2, G)$.

S is a well-order relation in $\mathbb{N}^2$.

So we visualize the 'ordering' for all the elements of $\mathbb{N}^2$ with respect to the well-order relation S, through the 'chain of inequalities' below:

$$(0,0) \preceq_S (1,0) \preceq_S (0,1) \preceq_S (2,0) \preceq_S (1,1) \preceq_S (3,0) \preceq_S (0,2) \preceq_S (2,1) \preceq_S (4,0) \preceq_S (1,2) \preceq_S (3,1) \preceq_S (0,3) \preceq_S \cdots$$

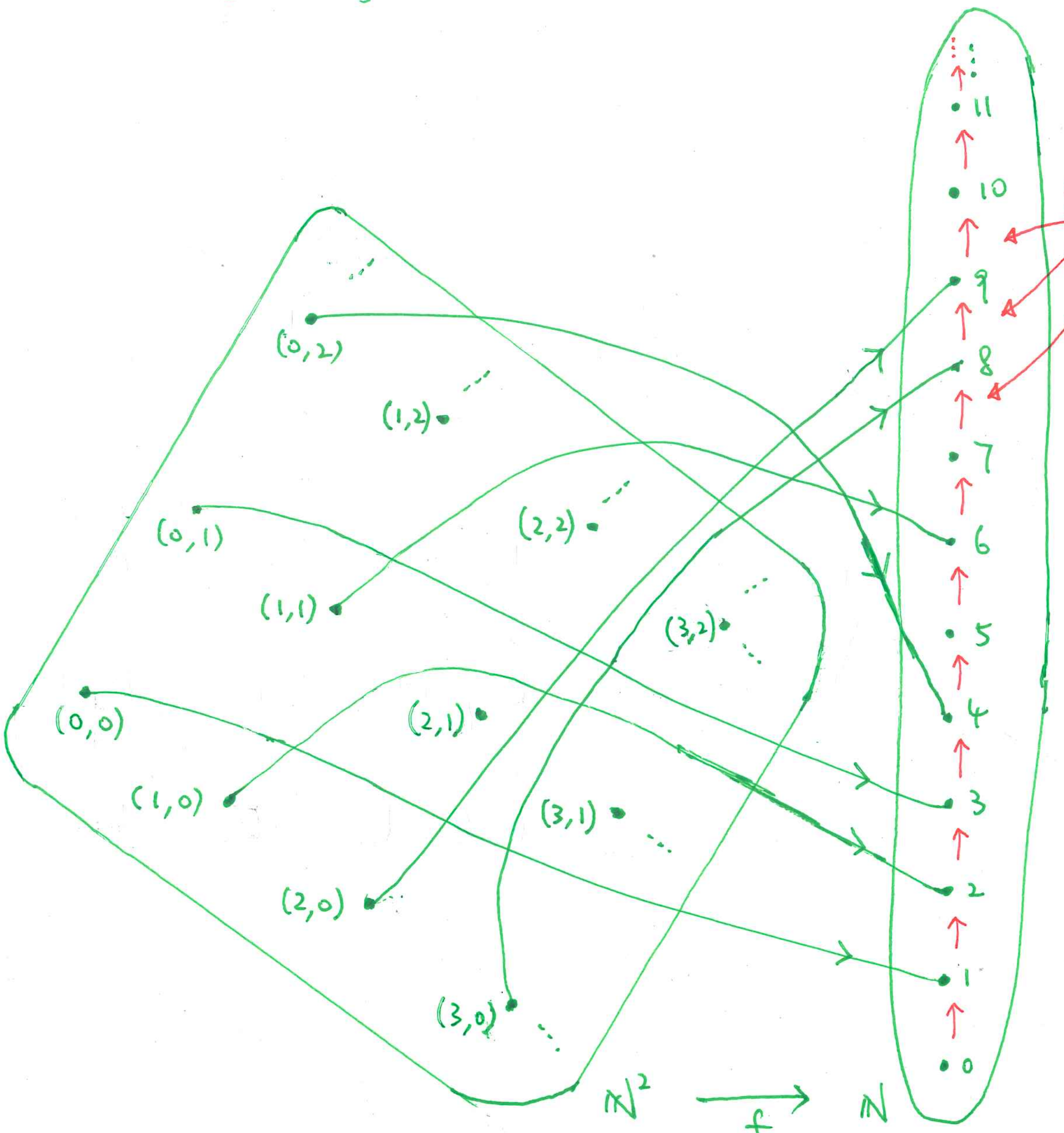
This is simply a direct translation, via f and S, of the chain of inequalities

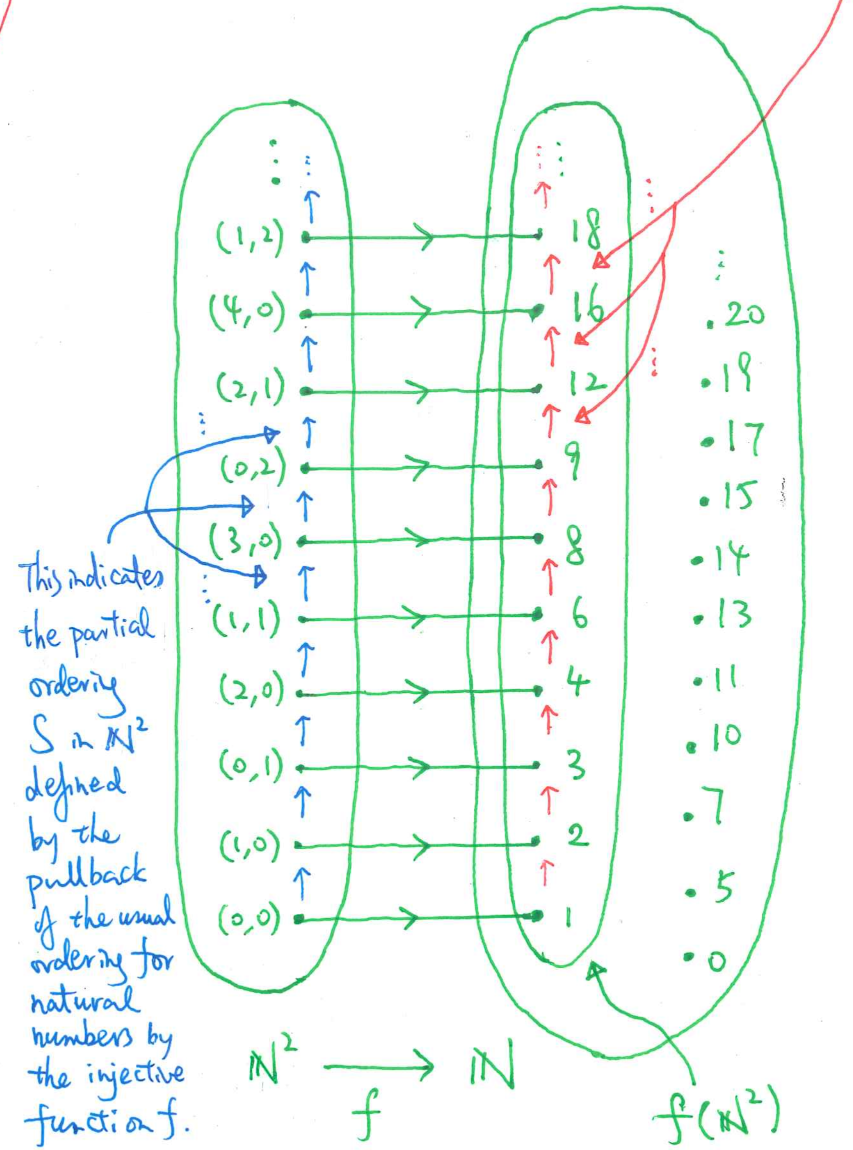
$$1 \leq 2 \leq 3 \leq 4 \leq 6 \leq 8 \leq 9 \leq 12 \leq 16 \leq 18 \leq 24 \leq 27 \leq \cdots .$$

Visualization for Example (F).

The injective function $f: \mathbb{N}^2 \to \mathbb{N}$ is given by $f(x, y) = 2^x \cdot 3^y$ for any $x, y \in \mathbb{N}$.

This indicates the usual ordering for natural numbers and its restriction to $f(\mathbb{N})$.





Note that

$$(1,0) \prec_S (0,1) \prec_S (2,0)$$

whereas

$$(1,0) <_{\text{lex}} (2,0) <_{\text{lex}} (0,1).$$

Hence S is certainly distinct from the lexicographical ordering for $\mathbb{N}^2$.

Remark. Replacing f by another injective function from $\mathbb{N}^2$ to $\mathbb{N}$, we will obtain another well-order relation in $\mathbb{N}^2$ from such a construction.

(For instance, what do you obtain with the injective function $g : \mathbb{N}^2 \longrightarrow \mathbb{N}$ given by $g(x,y) = 2^x 5^y$ for any $x, y \in \mathbb{N}$? Or how about the injective function $h : \mathbb{N}^2 \longrightarrow \mathbb{N}$ given by $h(x,y) = 3^x 5^y$ for any $x, y \in \mathbb{N}$?)

Example (E), Example (F) are illustrations of the idea in Theorem (12), which is concerned with general partial orderings.

26. **Theorem (12).**

Let A, B be sets, and $f : A \longrightarrow B$ be an injective function.

Suppose T is a partial ordering in B with graph H. Write $u \preceq_T v$ exactly when $(u, v) \in H$.

Define $G = \{(x, y) \mid x, y \in A \text{ and } f(x) \preceq_T f(y)\}$, and $S = (A, A, G)$.

Then S is a partial ordering in A with graph G.

If T is a total ordering in B then S is a total ordering in A.

If T is a well-order relation in B then S is a well-order relation in A.

Remark on terminology and notation. In the context of Theorem (12), the partial ordering S defined by the injective function f and the partial ordering T is called the **partial ordering in A defined by the pullback of T by f**. It is denoted by f^*T, and its graph is denoted by f^*H.

27. **Example (G). (Well-order relation in $\mathbb{Q}$ arising from the usual ordering for natural numbers.)**

Recall that that $\mathbb{Q}$ is not well-ordered by the usual ordering for integers.

However, we may define a well-order relation in $\mathbb{Q}$ with the help of the usual ordering for natural numbers.

(a) Refer to Example (E). We have constructed a well-order relation in $\mathbb{Z}$, namely, S, (with the help of the usual ordering for natural numbers).

(b) By Theorem (11), $\mathbb{Z}^2$ is well-ordered by the lexicographical ordering in $\mathbb{Z}^2$ induced by S and S.

We denote this well-order relation in $\mathbb{Z}^2$ by T.

S gives:

$0 \preceq_S -1 \preceq_S 1 \preceq_S -2 \preceq_S 2 \preceq_S -3 \preceq_S 3 \preceq_S \cdots$

T gives:

$(0,0) \preceq_T (0,-1) \preceq_T (0,1) \preceq_T (0,-2) \preceq_T (0,2) \preceq_T (0,-3) \preceq_T (0,3) \preceq_T \cdots$

$\preceq_T (-1,0) \preceq_T (-1,-1) \preceq_T (-1,1) \preceq_T (-1,-2) \preceq_T (-1,2) \preceq_T (-1,-3) \preceq_T (-1,3) \preceq_T \cdots$

$\preceq_T (1,0) \preceq_T (1,-1) \preceq_T (1,1) \preceq_T (1,-2) \preceq_T (1,2) \preceq_T (1,-3) \preceq_T (1,3) \preceq_T \cdots$

$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$

(c) We take the statement ($\sharp$) for granted:

($\sharp$) *For any* $r \in \mathbb{Q}\backslash\{0\}$, *there exist some unique* $p_r, q_r \in \mathbb{Z}$ *such that* $\gcd(p_r, q_r) = 1$ *and* $q_r > 0$ *and* $r = \dfrac{p_r}{q_r}$.

(Justify the statement ($\sharp$) as an exercise.)

Define the function $f : \mathbb{Q} \longrightarrow \mathbb{Z}^2$ by

$$f(r) = \begin{cases} (p_r, q_r) & \text{if} \quad r \in \mathbb{Q}\backslash\{0\} \\ (0, 1) & \text{if} \quad r = 0. \end{cases}$$

f is injective.

(d) According to Theorem (12), the partial ordering f^*T in $\mathbb{Q}$ defined by the pullback of T by f is a well-order relation in $\mathbb{Q}$.

How to visualize f^*T?

For any $u, v \in \mathbb{Q}$, write $u \leq v$ exactly when $\underbrace{(u,v) \text{ belongs to the graph of } f^*T}_{f(u) \preceq_T f(v)}$.

$$0 = \frac{0}{1}$$

$$\preceq -1 = \frac{-1}{1} \preceq -\frac{1}{2} = \frac{-1}{2} \preceq -\frac{1}{3} = \frac{-1}{3} \preceq -\frac{1}{4} = \frac{-1}{4} \preceq -\frac{1}{5} = \frac{-1}{5} \preceq -\frac{1}{6} = \frac{-1}{6} \preceq -\frac{1}{7} = \frac{-1}{7} \preceq \cdots\cdots$$

$$\preceq 1 = \frac{1}{1} \preceq \frac{1}{2} \preceq \frac{1}{3} \preceq \frac{1}{4} \preceq \frac{1}{5} \preceq \frac{1}{6} \preceq \frac{1}{7} \preceq \cdots\cdots$$

$$\preceq -2 = \frac{-2}{1} \preceq -\frac{2}{3} = \frac{-2}{3} \preceq -\frac{2}{5} = \frac{-2}{5} \preceq -\frac{2}{7} = \frac{-2}{7} \preceq \cdots\cdots$$

$$\preceq 2 = \frac{2}{1} \preceq \frac{2}{3} \preceq \frac{2}{5} \preceq \frac{2}{7} \preceq \cdots\cdots$$

$$\preceq -3 = \frac{-3}{1} \preceq -\frac{3}{2} = \frac{-3}{2} \preceq -\frac{3}{4} = \frac{-3}{4} \preceq -\frac{3}{5} = \frac{-3}{5} \preceq -\frac{3}{7} = \frac{-3}{7} \preceq \cdots\cdots$$

$$\preceq 3 = \frac{3}{1} \preceq \frac{3}{2} \preceq \frac{3}{4} \preceq \frac{3}{5} \preceq \frac{3}{7} \preceq \cdots\cdots$$

$$\vdots$$

Reminder. f^*T is not the usual ordering for rational numbers.

28. **Well-ordering Principle, as a fundamental assumption in mathematics.**

Example (E) and Example (G) tell us:

- Despite the fact that $\mathbb{Z}, \mathbb{Q}$ themselves are not well-ordered by the usual ordering for real numbers, it is still possible to equip them with various well-order relations.

We may ask: *Can we do the same thing for* $\mathbb{R}$*?*

If $\mathbb{R}$ can be equipped with a well-order relation, say, T, then the lexicographical ordering in $\mathbb{R}^2$ induced by T will be a well-order relation in $\mathbb{R}^2$, and will further provide a well-order relation for $\mathbb{C}$.

We may further ask: *Is it possible to equip any arbitrary set equipped with a well-order relation?*

It turns out that the answers to these questions are not quite trivial.

Well-ordering Principle.

Suppose A is a set. Then there exists some partial ordering T in A such that A is well-ordered by T.

Remark. We do not 'prove' the Well-ordering Principle. It is taken as a fundamental assumption in mathematics. (Of course, it is legitimate to choose between '*believing*' the Well-ordering Principle and '*not believing*' it.)