Chapter 9: Determinant

9.1 Definition

Suppose A is an $m \times n$ matrix. Then the submatrix $A(i|j)$ is the $(m-1) \times (n-1)$ matrix obtained from A by removing row i and column j.

Example 9.1.1: Suppose

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix}.$$

Then

$$A(2|3) = \begin{bmatrix} 1 & 2 & 4 \\ 9 & 10 & 12 \end{bmatrix}, \quad A(3|1) = \begin{bmatrix} 2 & 3 & 4 \\ 6 & 7 & 8 \end{bmatrix}.$$

■

Example 9.1.2: Suppose

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}.$$

Then

$$A(3|2) = \begin{bmatrix} a_{11} & a_{13} & a_{14} \\ a_{21} & a_{23} & a_{24} \\ a_{41} & a_{42} & a_{44} \end{bmatrix}, \quad A(4|1) = \begin{bmatrix} a_{12} & a_{13} & a_{14} \\ a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \end{bmatrix}.$$

■

The **determinant** is a function that take a square matrix as an input and produces a scalar as an output.

Definition 9.1.1: Suppose A is a square matrix. Then its *determinant*, $\det(A)$ (or denoted by $|A|$), is an element of $\mathbb{R}$ defined recursively by:

1. If A is a 1×1 matrix, then $\det(A) = [A]_{11}$.
2. If A is a matrix of order n with $n \geq 2$, then

$$\begin{aligned} \det(A) &= [A]_{1,1} \det(A\,(1|1)) - [A]_{1,2} \det(A\,(1|2)) + [A]_{1,3} \det(A\,(1|3)) + \\ &\quad + \cdots + (-1)^{n+1} [A]_{1,n} \det(A\,(1|n)) \\ &= \sum_{i=1}^{n} (-1)^{i+1} [A]_{1,i} \det(A\,(1|i)). \end{aligned}$$

So to compute the determinant of a 5×5 matrix we must build 5 submatrices, each of size 4. To compute the determinants of each the 4×4 matrices we need to create 4 submatrices each, these now of size 3 and so on. To compute the determinant of a 10×10 matrix would require computing the determinant of $10! = 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 = 3,628,800$ 1×1 matrices. Fortunately there are **better ways**.

Let us compute the determinant of a reasonably sized matrix by hand.

Example 9.1.3: Suppose that we have the 3×3 matrix

$$A = \begin{pmatrix} 3 & 2 & -1 \\ 4 & 1 & 6 \\ -3 & -1 & 2 \end{pmatrix}.$$

Proposition 9.1.2: *Suppose*

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Then

$$\det(A) = ad - bc.$$

Proof:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = a\det(d) - b\det(c) = ad - bc.$$

□

Proposition 9.1.3: *Suppose*

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

Then

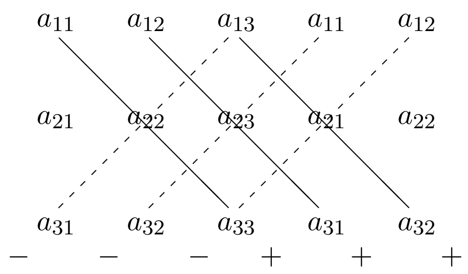
$$\det(A) = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}.$$

Proof:

$$\begin{aligned}
\det(A) &= a_{11}\det(A\,(1|1)) - a_{12}\det(A\,(1|2)) + a_{13}\det(A\,(1|3)) \\
&= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\
&= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}) \\
&= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}.
\end{aligned}$$

□

The *rule of Sarrus* is useful for memorizing the determinant of order 3:



Remark: The rule of Sarrus is valid ONLY for determinants of order 3.

9.2 Computing Determinants

Theorem 9.2.1: *Suppose that A is a square matrix of size n. For $1 \le i \le n$*

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} [A]_{i,j} \det(A(i|j)),$$

which is known as expansion about row *i.*

Skip the proof. If you are interested, see Beezer, p.266.

Theorem 9.2.2: *Suppose that A is a square matrix. Then* $\det(A^t) = \det(A)$.

Skip the proof. If you are interested, see Beezer, p.267.

Theorem 9.2.3: *Suppose that A is a square matrix of size n. Then for $1 \le j \le n$*

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+j} [A]_{i,j} \det(A(i|j)),$$

which is known as expansion about column *j.*

Follows from Theorems 9.2.1 and 9.2.2.

Example 9.2.1: Let

$$A = \begin{pmatrix} -2 & 3 & 0 & 1 \\ 9 & -2 & 0 & 1 \\ 1 & 3 & -2 & -1 \\ 4 & 1 & 2 & 6 \end{pmatrix}.$$

Then expanding about the fourth row yields,

$$\begin{aligned} |A| &= (-1)^{4+1}(4) \begin{vmatrix} 3 & 0 & 1 \\ -2 & 0 & 1 \\ 3 & -2 & -1 \end{vmatrix} + (-1)^{4+2}(1) \begin{vmatrix} -2 & 0 & 1 \\ 9 & 0 & 1 \\ 1 & -2 & -1 \end{vmatrix} \\ &\quad + (-1)^{4+3}(2) \begin{vmatrix} -2 & 3 & 1 \\ 9 & -2 & 1 \\ 1 & 3 & -1 \end{vmatrix} + (-1)^{4+4}(6) \begin{vmatrix} -2 & 3 & 0 \\ 9 & -2 & 0 \\ 1 & 3 & -2 \end{vmatrix} \\ &= (-4)(10) + (1)(-22) + (-2)(61) + 6(46) = 92 \end{aligned}$$

Example 9.2.2: Suppose that

$$U = \begin{pmatrix} 2 & 3 & -1 & 3 & 3 \\ 0 & -1 & 5 & 2 & -1 \\ 0 & 0 & 3 & 9 & 2 \\ 0 & 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 0 & 5 \end{pmatrix}.$$

We will compute the determinant of this 5×5 matrix by consistently expanding about the first column for each submatrix that arises and does not have a zero entry multiplying it.

Theorem 9.2.4: *Suppose A is an upper triangular matrix, i.e.,*

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix}.$$

Then

$$\det(A) = a_{11}a_{22}\cdots a_{nn}.$$

Proof:

$$\begin{aligned}
\det(A) &= a_{11}\det\begin{bmatrix} a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn}\end{bmatrix} && \text{expand along the column 1} \\
&= a_{11}a_{22}\det\begin{bmatrix} a_{33} & \cdots & a_{3n} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a_{nn}\end{bmatrix} && \text{expand along the column 1} \\
&\cdots \\
&= a_{11}a_{22}\cdots a_{nn}.
\end{aligned}$$

□

Similarly

Theorem 9.2.5: *Suppose A is a lower triangular matrix, i.e.,*

$$A = \begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & \cdots & 0 \\ a_{31} & a_{32} & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn}\end{bmatrix}.$$

Then

$$\det(A) = a_{11}a_{22}\cdots a_{nn}.$$

When you consult other texts in your study of determinants, you may run into the terms **minor** and **cofactor**, especially in a discussion centered on expansion about rows and columns. We have chosen not to make these definitions formally since we have been able to get along without them. However, informally, a *minor* is a determinant of a submatrix, specifically $\det(A(i|j))$ and is usually referenced as the minor of $[A]_{ij}$. A *cofactor* is a signed minor, specifically the cofactor of $[A]_{ij}$ is $(-1)^{i+j}\det(A(i|j))$.

9.3 Properties of Determinants of Matrices

Theorem 9.3.1: *Suppose that A is a square matrix with a row where every entry is zero, or a column where every entry is zero. Then* $\det(A) = 0$.

Proof: Suppose that A is a square matrix of size n and row i has every entry equal to zero. We compute $\det(A)$ via expansion about row i.

$$\begin{aligned}
\det(A) &= \sum_{j=1}^{n}(-1)^{i+j}[A]_{ij}\det(A(i|j)) = \sum_{j=1}^{n}(-1)^{i+j}(0)\det(A(i|j)) && \text{Row } i \text{ is zeros} \\
&= \sum_{j=1}^{n} 0 = 0.
\end{aligned}$$

The proof for the case of a zero column is entirely similar, or could be derived by the fact that $\det(A) = \det(A^t)$. □

By means of Theorem 9.2.2, the proofs of the following theorems are only focused on rows.

Theorem 9.3.2: *Suppose that A is a square matrix. Let B be the matrix obtained from A by interchanging the location of two rows, or interchanging the location of two columns, i.e., $A \xrightarrow{\mathcal{R}_i \leftrightarrow \mathcal{R}_j} B$ or $A \xrightarrow{\mathcal{C}_i \leftrightarrow \mathcal{C}_j} B$ for some $i \neq j$. Then* $\det(A) = -\det(B)$.

Skip the proof. If you are interested, see Beezer p.273.

Theorem 9.3.3: *Suppose that A is a square matrix with two equal rows, or two equal columns. Then* $\det(A) = 0$.

Theorem 9.3.4: *Suppose that A is a square matrix. Let B be the square matrix obtained from A by multiplying a single row (say, row i) by the scalar c, or by multiplying a single column by the scalar c, i.e., $A \xrightarrow{c\mathcal{R}_i} B$ or $A \xrightarrow{c\mathcal{C}_i} B$. Then* $\det(B) = c\det(A)$, *i.e.,* $\det(A) = c^{-1}\det(B)$.

Proof:

Theorem 9.3.5: *Suppose that $A \in M_n$. Let B be the matrix obtained from A by multiplying the row i by a scalar c and adding to row j (or by multiplying the column i by a scalar c and adding to the column j), $i \neq j$. Then* $\det(B) = \det(A)$.

Proof:

Theorem 9.3.6:

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{i-1,1} & a_{i-1,2} & \cdots & a_{i-1,n} \\ b_1+c_1 & b_2+c_2 & \cdots & b_n+c_n \\ a_{i+1,1} & a_{i+1,2} & \cdots & a_{i+1,n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{i-1,1} & a_{i-1,2} & \cdots & a_{i-1,n} \\ b_1 & b_2 & \cdots & b_n \\ a_{i+1,1} & a_{i+1,2} & \cdots & a_{i+1,n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{i-1,1} & a_{i-1,2} & \cdots & a_{i-1,n} \\ c_1 & c_2 & \cdots & c_n \\ a_{i+1,1} & a_{i+1,2} & \cdots & a_{i+1,n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}.$$

Similarly

$$\begin{vmatrix} a_{11} & \cdots & a_{1,j-1} & b_1+c_1 & a_{1,j+1} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2,j-1} & b_2+c_2 & a_{2,j+1} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{n,j-1} & b_n+c_n & a_{n,j+1} & \cdots & a_{nn} \end{vmatrix}$$
$$= \begin{vmatrix} a_{11} & \cdots & a_{1,j-1} & b_1 & a_{1,j+1} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2,j-1} & b_2 & a_{2,j+1} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{n,j-1} & b_n & a_{n,j+1} & \cdots & a_{nn} \end{vmatrix} + \begin{vmatrix} a_{11} & \cdots & a_{1,j-1} & c_1 & a_{1,j+1} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2,j-1} & c_2 & a_{2,j+1} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{n,j-1} & c_n & a_{n,j+1} & \cdots & a_{nn} \end{vmatrix}.$$

Proof: Expand along row i (or column j). □

We will perform a sequence of elementary row operations on a matrix, shooting for an upper triangular matrix, whose determinant will be simply the product of its diagonal entries. For each row operation, we will track the effect on the determinant via Theorems 9.3.4, 9.3.5 and 9.3.2.

Example 9.3.1: Compute

$$\begin{vmatrix} 2 & 0 & 2 & 3 \\ 1 & 3 & -1 & 1 \\ -1 & 1 & -1 & 2 \\ 3 & 5 & 4 & 0 \end{vmatrix}.$$

9.4 Examples

Example 9.4.1: Compute

$$\begin{vmatrix} 1 & a_1 & a_2 & a_3 \\ 1 & a_1+b_1 & a_2 & a_3 \\ 1 & a_1 & a_2+b_2 & a_3 \\ 1 & a_1 & a_2 & a_3+b_3. \end{vmatrix}.$$

Example 9.4.2: Let A_n be an $n \times n$ matrix

$$\underbrace{\begin{pmatrix} x & 1 & 1 & \cdots & 1 \\ 1 & x & 1 & \cdots & 1 \\ 1 & 1 & x & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & 1 \\ 1 & 1 & 1 & \cdots & x \end{pmatrix}}_{n} \left.\vphantom{\begin{pmatrix} x \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}}\right\} n$$

Find $\det(A_n)$.

Example 9.4.3: Let B_n be an $n \times n$ matrix in the form

$$\begin{bmatrix} 1-a_1 & a_2 & 0 & \cdots & 0 & 0 \\ -1 & 1-a_2 & a_3 & \cdots & 0 & 0 \\ 0 & -1 & 1-a_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1-a_{n-1} & a_n \\ 0 & 0 & 0 & \cdots & -1 & 1-a_n \end{bmatrix}.$$

1. Show that $\det(B_n) = \det(B_{n-1}) + (-1)^n(a_1a_2\cdots a_n)$.

2. Hence show $\det(B_n) = 1 + \sum_{i=1}^{n}(-1)^i(a_1a_2\cdots a_i)$.

Answer:

1. By adding $\mathcal{R}_1, \ldots, \mathcal{R}_{n-1}$ to $\mathcal{R}_n$, we have

$$\det(B_n) = \begin{vmatrix} 1-a_1 & a_2 & 0 & \cdots & 0 & 0 \\ -1 & 1-a_2 & a_3 & \cdots & 0 & 0 \\ 0 & -1 & 1-a_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1-a_{n-1} & a_n \\ -a_1 & 0 & 0 & \cdots & 0 & 1 \end{vmatrix}.$$

Expand along the last row, the above determinant is

$$(-1)^{n+1}(-a_1)\begin{vmatrix} a_2 & 0 & \cdots & 0 & 0 \\ 1-a_2 & a_3 & \cdots & 0 & 0 \\ -1 & 1-a_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1-a_{n-1} & a_n \end{vmatrix} + (-1)^{n+n}\begin{vmatrix} 1-a_1 & a_2 & 0 & \cdots & 0 \\ -1 & 1-a_2 & a_3 & \cdots & 0 \\ 0 & -1 & 1-a_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1-a_{n-1} \end{vmatrix}.$$

The first matrix is a lower triangular matrix, so the determinant is the product of the diagonal entries, the second matrix is B_{n-1}. Thus,

$$\det(B_n) = (-1)^n(a_1\cdots a_n) + \det(B_{n-1}).$$

2. We prove the result by mathematical induction:

Step 1: The formula is valid for $n = 1$: $\det(B_1) = 1 - a_1$.

Step 2: Suppose the formula is true for $n = k$, we want to show that the formula is true for $n = k+1$:

$$\begin{aligned}\det(B_{k+1}) &= (-1)^{k+1}(a_1\cdots a_{k+1}) + \det(B_k) \\ &= 1 + \sum_{i=1}^{k}(-1)^i(a_1a_2\cdots a_i) + (-1)^{k+1}(a_1\cdots a_{k+1}) \\ &= 1 + \sum_{i=1}^{k+1}(-1)^i(a_1\cdots a_i).\end{aligned}$$

The formula is true for $n = k+1$.

Step 3: By mathematical induction, the formula is valid for all positive integer.

Explanation: The formula is true for $k = 1$, then it is true for $k+1 = 2$, so true for $k+1 = 3$, etc. Hence true for all integers. This process is called **mathematical induction**.

Example 9.4.4: Let C_n be an $n \times n$ matrix given by

$$C_n = \underbrace{\begin{pmatrix} x & a & a & \cdots & a & a \\ -a & x & a & \cdots & a & a \\ -a & -a & x & \cdots & a & a \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -a & -a & -a & \cdots & x & a \\ -a & -a & -a & \cdots & -a & x \end{pmatrix}}_{n} \left.\vphantom{\begin{pmatrix} x \\ x \\ x \\ x \\ x \\ x \end{pmatrix}}\right\} n$$

1. Show that $\det(C_n) = a(x+a)^{n-1} + (x-a)\det(C_{n-1})$.

2. Show that $\det(C_n) = \frac{1}{2}((x+a)^n + (x-a)^n)$.

Leave to students as an exercise.

Example 9.4.5: (**Vandermonde Determinant**) This is the most important example of determinant. Let

$$V_n = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ a_1 & a_2 & a_3 & \cdots & a_n \\ a_1^2 & a_2^2 & a_3^2 & \cdots & a_n^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_1^{n-2} & a_2^{n-2} & a_3^{n-2} & \cdots & a_n^{n-2} \\ a_1^{n-1} & a_2^{n-1} & a_3^{n-1} & \cdots & a_n^{n-1} \end{bmatrix},$$

where $n \geq 2$.

1. $\det(V_n) = \det(V_{n-1}) \prod_{i=1}^{n-1} (a_n - a_i)$.

2. $\det(V_n) = \prod_{1 \leq i < j \leq n} (a_j - a_i)$.

Answer:

1. Applying $-a_n\mathcal{R}_{n-1} + \mathcal{R}_n$, $-a_n\mathcal{R}_{n-2} + \mathcal{R}_{n-1}$, $\ldots$, $-a_n\mathcal{R}_1 + \mathcal{R}_2$, we have

$$\det(V_n) = \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ a_1 - a_n & a_2 - a_n & a_3 - a_n & \cdots & 0 \\ a_1^2 - a_1a_n & a_2^2 - a_2a_n & a_3^2 - a_3a_n & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_1^{n-2} - a_1^{n-3}a_n & a_2^{n-2} - a_2^{n-3}a_n & a_3^{n-2} - a_3^{n-3}a_n & \cdots & 0 \\ a_1^{n-1} - a_1^{n-2}a_n & a_2^{n-1} - a_2^{n-2}a_n & a_3^{n-1} - a_3^{n-2}a_n & \cdots & 0 \end{vmatrix}.$$

2.

9.5 More Properties of Determinants

By Theorems 9.3.4, 9.3.5 and 9.3.2 we have

Corollary 9.5.1:

1. Let $I_n \xrightarrow{\mathcal{R}_i \leftrightarrow \mathcal{R}_j} E$*,* $i \neq j$*. Then* $\det(E) = -1$*.*

2. Let $I_n \xrightarrow{c\mathcal{R}_i} E$*. Then* $\det(E) = c$*.*

3. Let $I_n \xrightarrow{c\mathcal{R}_i+\mathcal{R}_j} E$, $i \neq j$. *Then* $\det(E) = 1$.

Corollary 9.5.2: *Let A be a square matrix. Let B be the matrix obtained from A by applying an elementary row operation on A. Let E be obtained by applying the same row operation on* I_n. *Then* $B = EA$ *and* $\det(B) = \det(EA) = \det(E)\det(A)$.

Proof: By Theorem 5.4.4, $B = EA$. The last result follows from Theorems 9.3.4, 9.3.5 and 9.3.2 and Corollary 9.5.1. □

Theorem 9.5.3: *A is nonsingular if and only if* $\det(A) \neq 0$.

Proof: Suppose A is nonsingular. By Theorem 8.3.7 A is row equivalent to I_n. By Corollary 3.1.4, $A = PI_n = P$, where P is a product of elementary matrices. Applying Corollaries 9.5.2 and 9.5.1 repeatedly we have $\det(A) \neq 0$.

Suppose $A \in M_n$ is singular. By Theorem 5.4.4 and 8.3.7, $PA = \text{rref}(A)$ with $\text{rank}(A) < n$, where P is a product of elementary matrices. Hence rref(A) has a zero row. Therefore $\det(PA) = 0$. Applying Corollary 9.5.2 repeatedly, $\det(PA) = \det(P)\det(A)$. Since P is a product of elementary matrices (invertible), $\det(P) \neq 0$. So $\det(A) = 0$. □

In fact we have the following stronger result:

Theorem 9.5.4: *Suppose* $A, B \in M_n$. *Then*

$$\det(AB) = \det(A)\det(B).$$

Proof: Suppose A is nonsingular. By Theorem 8.3.7, A is row equivalent to I_n. By Theorem 5.4.4, A is a product of elementary matrices. Applying Corollary 9.5.2 repeatedly we have $\det(AB) = \det(A)\det(B)$.

Suppose A is singular. By Theorem 5.3.3. AB is singular. By Theorem 9.5.3, $\det(A) = 0$ and $\det(AB) = 0$. Hence $\det(AB) = \det(A)\det(B) = 0$.

□

Corollary 9.5.5: *If A is invertible, then* $\det(A^{-1}) = [\det(A)]^{-1}$.

Theorem 9.5.6 (Cramer's rule)**:** *Let A be an invertible matrix of order n. Let* $\boldsymbol{b} \in \mathbb{R}^n$. *Let* M_k *be the square matrix by replacing the k-th column of A by* $\boldsymbol{b}$. *If*

$$\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^t$$

is a solution of $A\boldsymbol{x} = \boldsymbol{b}$, *then*

$$x_k = \frac{\det(M_k)}{\det(A)}$$

where $k = 1, \ldots, n$.

Proof:

Example 9.5.1: Using Carmer's rule to solve the following system of linear equation.

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & + & 3x_3 & = & 2 \\ x_1 & & & + & x_3 & = & 3 \\ x_1 & + & x_2 & - & x_3 & = & 1 \end{array}$$

Solution:

Theorem 9.5.7 (Formula for inverse): *Suppose $A \in M_n$ is an invertible matrix. Then*

$$[A^{-1}]_{ji} = \frac{(-1)^{i+j}\det(A(i|j))}{\det(A)}.$$

Pay attention to the order of the indices i and j!

Proof: For convenience, write $B = A^{-1}$. That is, $AB = I_n$.

Let C be the square matrix of order n whose (i, j)-the entry is defined by $(-1)^{i+j}\det(A(i|j))$, i.e., $[C]_{ij} = (-1)^{i+j}\det(A(i|j)) = \det(A)[B]_{ji}$.

In many textbooks, the matrix C is called the *cofactor matrix* of A. The transpose of C, C^t, is called the *adjoint matrix* of A which is denoted by $\mathrm{adj}(A)$. So $\mathrm{adj}(A) = C^t = \det(A)B$ and we have

$$A^{-1} = B = \frac{1}{\det(A)}\mathrm{adj}(A).$$

In general, the formula

$$A\mathrm{adj}(A) = \det(A)I_n = \mathrm{adj}(A)A$$

always holds.

Example 9.5.2: By the above formula, find the inverse of

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 1 & 1 & -1 \end{pmatrix}.$$

Solution: Firstly we have $\det(A) = 6$.

$$A(1|1) = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}, \quad \det(A(1|1)) = -1; \quad A(1|2) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad \det(A(1|2)) = -2;$$

$$A(1|3) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad \det(A(1|3)) = 1; \quad A(2|1) = \begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix}, \quad \det(A(2|1)) = -5;$$

$$A(2|2) = \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix}, \quad \det(A(2|2)) = -4; \quad A(2|3) = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}, \quad \det(A(2|3)) = -1;$$

$$A(3|1) = \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}, \quad \det(A(3|1)) = 2; \quad A(3|2) = \begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix}, \quad \det(A(3|2)) = -2;$$

$$A(3|3) = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}, \quad \det(A(3|3)) = -2.$$