Chapter 3: Gaussian Elimination

3.1 Elementary Row Operations

Definition 3.1.1: The following three operations will transform an $m \times n$ matrix into a different matrix of the same size, and each is called an *elementary row operation*.

Type 1: Multiply the i-th row by a nonzero scalar b. We use "$b\mathcal{R}_i$" to denote this operation.

Type 2: Multiply the i-th row by b and add to the j-th row for $i \neq j$. We denote this operation by the notation "$b\mathcal{R}_i + \mathcal{R}_j$". Note that the i-th row does not change under this operation.

Type 3: Interchange the i-th and the j-th rows for $i \neq j$. We use "$\mathcal{R}_i \leftrightarrow R_j$" to denote this operation.

Definition 3.1.2: An *elementary matrix* is a matrix obtained by applying an elementary row operation to the identity matrix. The elementary matrix is said to be of *type* 1, 2, or 3 according to whether the elementary operation of type 1, 2 or 3 performed on I, respectively.

For fixed integers i, j, let $E^{i,j} \in M_m(\mathbb{R})$ whose (i,j)-entry is 1 and others are zero, where m fixed, i.e., $[E^{i,j}]_{h,k} = \delta_{hi}\delta_{kj}$.

So an elementary matrix of type 1 is $(b\mathcal{R}_i)(I) = I + (b-1)E^{i,i}$ for $b \neq 0$ having the form

$$\begin{array}{c} \\ 1 \\ 2 \\ \\ i \\ \\ n \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & & i & & n \end{array} \\ \begin{pmatrix} 1 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & 1 & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & b & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & 1 \end{pmatrix} \end{array}$$

An elementary matrix of type 2 is $(b\mathcal{R}_i + \mathcal{R}_j)(I) = I + bE^{j,i}$ for $i < j$ and $b \neq 0$ having the form

$$\begin{array}{c} \\ 1 \\ 2 \\ \\ i \\ \\ j \\ \\ n \end{array} \begin{array}{c} \begin{array}{cccccccc} 1 & 2 & & i & & j & & n \end{array} \\ \begin{pmatrix} 1 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & 1 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & b & \cdots & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 \end{pmatrix} \end{array}$$

An elementary matrix of type 3 is $(\mathcal{R}_i \leftrightarrow \mathcal{R}_j)(I) = I - E^{i,i} - E^{j,j} + E^{i,j} + E^{j,i}$ for $i < j$ having the form

$$\begin{array}{c} \\ 1 \\ 2 \\ \\ i \\ \\ j \\ \\ n \end{array}\begin{array}{c} \begin{array}{cccccccc} 1 & 2 & & i & & j & & n \end{array} \\ \begin{pmatrix} 1 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & 0 & \cdots & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & 1 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 \end{pmatrix} \end{array}$$

Theorem 3.1.3: *Let $A \in M_{m,n}$ and suppose that B is obtained from A by performing an elementary row operation. Then there is an $m \times m$ elementary matrix E such that $B = EA$. In fact, E is obtained by performing the same row operation on I_m. Conversely, if E is an $m \times m$ elementary matrix, then EA is a matrix that can be obtained by performing the same elementary row operation on A.*

Corollary 3.1.4: *Suppose $A \in M_{m,n}$. After performing s elementary row operations we obtain B. By Theorem 3.1.3, there are s elementary matrices $E_1, \ldots, E_s$ such that*

$$B = E_s \cdots E_1 A = PA,$$

here $P = E_s \cdots E_1$, a product of elementary matrices.

How do we record the product of elementary matrices described in the corollary above?

We may consider a block matrix $(A|I_m)$. If we perform a sequence of elementary row operations transferring A to B, then by the above corollary there is a matrix P such that $PA = B$. Hence $P(A|I_m) = (B|P)$.

3.2 Reduced Row Echelon Form

Terminologies:
A row consists of 0 is called a *zero row.* Otherwise is called a nonzero row.
The first nonzero entry of a nonzero row is called *leftmost nonzero entry of a row.*
Suppose the row i is nonzero. The index of the leftmost nonzero entry of this row is denoted by d_i.

Example 3.2.1: The underline entry is the leftmost nonzero entry for each row.

$$\begin{bmatrix} 0 & \underline{1} & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & \underline{1} \\ 0 & 0 & 0 & \underline{1} & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The index of the leftmost nonzero entry of row 1 is $d_1 = 2$.
The index of the leftmost nonzero entry of row 2 is $d_2 = 5$.
The index of the leftmost nonzero entry of row 3 is $d_3 = 4$.
Row 4 is a zero row. ■

Example 3.2.2: The boxed entry is the leftmost nonzero entry for each row.

$$\begin{bmatrix} \boxed{2} & 0 & 1 & 2 & 3 & 4 \\ 0 & \boxed{1} & 1 & -1 & 0 & 3 \\ 0 & 0 & 0 & 0 & \boxed{1} & 0 \\ 0 & \boxed{-1} & 0 & 0 & 0 & 1 \end{bmatrix}$$

The index of the leftmost nonzero entry of row 1 is $d_1 = 1$.
The index of the leftmost nonzero entry of row 2 is $d_2 = 2$.
The index of the leftmost nonzero entry of row 3 is $d_3 = 5$.
The index of the leftmost nonzero entry of row 4 is $d_4 = 2$. ■

A matrix is said to be in reduced row echelon form if it looks like ($*$ means an arbitrary number):

$$\begin{bmatrix} ① & * & \cdots & 0 & * & \cdots & 0 & * & \cdots \\ 0 & 0 & \cdots & ① & * & \cdots & 0 & * & \cdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & ① & * & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

1. It looks like an inverted staircase.

2. Each new step down gives a ①. Above it are zeros.

3. The column that has a new step is called the pivot column.

Let us make the formal definition.

Definition 3.2.1: An $m \times n$ matrix is called in *reduced row echelon form* (*rref*) if it satisfies the following four conditions:

1. The zero rows, if any, are the last rows of the matrix.

2. The leftmost nonzero entry in a nonzero row is a 1. It is called a *pivot* or *leading one*.

3. In the d_i-th column, the only nonzero entry is the pivot in the i-th row.

4. Suppose there are r nonzero rows. Let the pivot appearing in the i-th row lie at (i, d_i)-entry, $1 \le i \le r$. Then $1 \le d_1 < d_2 < \cdots < d_r \le n$.

A column containing a pivot is called *leading column* (or *pivot column*).

Suppose there are r nonzero rows. Then there are exactly r leading columns and also r pivots. Here is a general reduced row echelon form:

$$
\begin{array}{cccccccccccc}
 & & & d_1 & & & d_i & & & d_r & & \\
\end{array}
$$

$$
\left(\begin{array}{cccccccccccc}
0 & \cdots & 0 & 1 & b_{1,d_1+1} & \cdots & 0 & b_{1,d_i+1} & \cdots & 0 & b_{1,d_r+1} & \cdots \\
0 & \cdots & 0 & 0 & * & \cdots & 0 & b_{2,d_i+1} & \cdots & 0 & b_{2,d_r+1} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & 0 & 0 & \cdots & \cdots & 1 & b_{i,d_i+1} & \cdots & 0 & b_{i,d_r+1} & \cdots \\
0 & \cdots & 0 & 0 & \cdots & \cdots & 0 & * & \cdots & 0 & b_{i+1,d_r+1} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & 0 & 0 & \cdots & \cdots & 0 & \cdots & \cdots & 0 & b_{r-1,d_r+1} & \cdots \\
0 & \cdots & 0 & 0 & \cdots & \cdots & 0 & \cdots & \cdots & 1 & b_{r,d_r+1} & \cdots \\
\hline
0 & \cdots & 0 & 0 & \cdots & \cdots & 0 & \cdots & \cdots & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & 0 & 0 & \cdots & \cdots & 0 & \cdots & \cdots & 0 & \cdots & 0
\end{array}\right)
$$

where $*$ is either 0 or 1. If $*$ is a pivot, then every entry in this column differing from it is zero.

Following matrices are in rref:

$$
I_n, \quad \mathcal{O}_{m,n}, \quad \begin{bmatrix} 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.
$$

Following matrices are not in reduced row echelon form.

$$
\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.
$$

The set of indices of columns that are leading columns will be denoted as $D = \{d_1, d_2, \ldots, d_r\}$, while that of columns which are not leading columns will be denoted as $F = \{f_1, f_2, \ldots, f_{n-r}\}$, where $1 \le f_1 < f_2 < \cdots < f_{n-r} \le n$.

Note that, the i-th leading column (i.e., the d_i-th column of the matrix) is $\boldsymbol{e}_i$ (of length m), $1 \le i \le r$.

Example 3.2.3: The 4×6 matrix below is in reduced row echelon from

$$
\begin{bmatrix} 1 & 3 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 3 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.
$$

■

Example 3.2.4: The following 4×7 matrix is in rref

$$
\begin{bmatrix} 1 & 0 & 5 & 3 & 0 & 0 & 5 \\ 0 & 1 & 3 & 6 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 & 1 & 0 & 7 \\ 0 & 0 & 0 & 0 & 0 & 1 & 3 \end{bmatrix}.
$$

■

Example 3.2.5:

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is in rref.

■

Example 3.2.6: Here is a 5×8 matrix in rref

$$\begin{pmatrix} \textcircled{1} & 1 & 0 & 6 & 0 & 0 & -5 & 9 \\ 0 & 0 & 0 & 0 & \textcircled{1} & 0 & 3 & -7 \\ 0 & 0 & 0 & 0 & 0 & \textcircled{1} & 7 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

■

Definition 3.2.2: Two matrices A and B are called *row equivalent* if there is a sequence of elementary row operations that transforms A to B. We use $A \sim B$ to denote that A and B are row equivalent. It is important to note that row operations are *reversible.* So $A \sim B$ implies $B \sim A$ and vice versa.

Theorem 3.2.3: *Each matrix is row equivalent to a reduced row echelon matrix.*

Proof: We may assume $A \neq \mathcal{O}$. The following algorithm is called *Gaussian elimination.* Using this algorithm we can reduce the matrix A to an rref matrix.

Gaussian Elimination (Gauss-Jordan Method)

Step 1. If the first column of the matrix is a zero column, cross it off mentally. Continue in this fashion until the left column of the remaining matrix has a nonzero entry or until the columns are exhausted. For the last case, go to Step 6.

Step 2. Interchange the first row with another row, if necessary, to put a nonzero entry to the top of the first column.

Step 3. By means of operation of type 1, make the nonzero entry found in Step 2 to be 1 (a pivot).

Step 4. By means of operations type 2 ($b\mathcal{R}_i + \mathcal{R}_j$), use the first row to obtain zeros in the remaining positions of the first column.

Step 5. Cross off the first row and the first column mentally. Begin with Step 1 applied to the submatrix that remains.

Step 6. Beginning with the last nonzero row, add multiples of this row to the rows above (operations type 2) such that the pivot in this row is the only nonzero entry in its column.

Step 7. Use operations type 2 to make the pivot in the next-to-last row the only nonzero entry in its column.

Step 8. Repeat Step 7 for each preceding row until the second row is performed. □

Example 3.2.7: Using the Gaussian elimination, find the rref of

$$A = \begin{pmatrix} 0 & 0 & 2 & 2 & 6 & 2 & 3 \\ 2 & 4 & 1 & 3 & 7 & 3 & -1 \\ 1 & 2 & 2 & 3 & 8 & 2 & 1 \\ 1 & 2 & -1 & 0 & -1 & 2 & -1 \end{pmatrix}.$$

Solution:

- We first work on row 1.
- Consider column 1, find a nonzero entry in the column.
- Move the nonzero entry to row 1 by swapping rows $\mathcal{R}_1 \leftrightarrow \mathcal{R}_i$ for some $i \neq 1$ (Step 2).
- If the $(1,1)$-entry is nonzero, you do not have to swap rows. But you can consider swap it with entry $= 1$ or -1.
- In this example, for column 1, 2nd entry, 3rd entry and 4th entry are nonzeros, so we can use $\mathcal{R}_1 \leftrightarrow \mathcal{R}_2$, $\mathcal{R}_1 \leftrightarrow \mathcal{R}_3$ or $\mathcal{R}_1 \leftrightarrow \mathcal{R}_4$.
- There is nothing wrong about $\mathcal{R}_1 \leftrightarrow \mathcal{R}_2$ but *it is better to swap with the row with entry equal to* 1 *or* -1.
- So we use $\mathcal{R}_1 \leftrightarrow \mathcal{R}_3$.

We have

$$A \xrightarrow{\mathcal{R}_1 \leftrightarrow \mathcal{R}_3} \begin{pmatrix} \textcircled{1} & 2 & 2 & 3 & 8 & 2 & 1 \\ 2 & 4 & 1 & 3 & 7 & 3 & -1 \\ 0 & 0 & 2 & 2 & 6 & 2 & 3 \\ 1 & 2 & -1 & 0 & -1 & 2 & -1 \end{pmatrix} = A_1$$

After Step 4, we have

$$A_1 \xrightarrow[-\mathcal{R}_1+\mathcal{R}_4]{-2\mathcal{R}_1+\mathcal{R}_2} \begin{pmatrix} \textcircled{1} & 2 & 2 & 3 & 8 & 2 & 1 \\ 0 & 0 & -3 & -3 & -9 & -1 & -3 \\ 0 & 0 & 2 & 2 & 6 & 2 & 3 \\ 0 & 0 & -3 & -3 & -9 & 0 & -2 \end{pmatrix} = A_3$$

Ignore the row 1 and column 1. No nonzero entry lies in the remaining column 2, so we move to next column (Step 1). Now, we only need to focus on the 'right-lower submatrix' of A_3 consider the matrix

$$A_3 = \left(\begin{array}{cc|ccccc} \textcircled{1} & 2 & 2 & 3 & 8 & 2 & 1 \\ \hline 0 & 0 & \boxed{-3} & -3 & -9 & -1 & -3 \\ 0 & 0 & 2 & 2 & 6 & 2 & 3 \\ 0 & 0 & -3 & -3 & -9 & 0 & -2 \end{array}\right)$$

$$A_3 \xrightarrow{-\frac{1}{3}\mathcal{R}_2} \left(\begin{array}{cc|ccccc} \textcircled{1} & 2 & 2 & 3 & 8 & 2 & 1 \\ \hline 0 & 0 & \textcircled{1} & 1 & 3 & \frac{1}{3} & 1 \\ 0 & 0 & 2 & 2 & 6 & 2 & 3 \\ 0 & 0 & -3 & -3 & -9 & 0 & -2 \end{array}\right) \xrightarrow[3\mathcal{R}_2+\mathcal{R}_4]{-2\mathcal{R}_2+\mathcal{R}_3} \left(\begin{array}{ccccc|cc} \textcircled{1} & 2 & 2 & 3 & 8 & 2 & 1 \\ 0 & 0 & \textcircled{1} & 1 & 3 & \frac{1}{3} & 1 \\ \hline 0 & 0 & 0 & 0 & 0 & \frac{4}{3} & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right)$$

$$\xrightarrow{\mathcal{R}_3\leftrightarrow\mathcal{R}_4} \left(\begin{array}{ccccc|cc} \textcircled{1} & 2 & 2 & 3 & 8 & 2 & 1 \\ 0 & 0 & \textcircled{1} & 1 & 3 & \frac{1}{3} & 1 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & \frac{4}{3} & 1 \end{array}\right) \xrightarrow{-\frac{4}{3}\mathcal{R}_3+\mathcal{R}_4} \left(\begin{array}{cccccc|c} \textcircled{1} & 2 & 2 & 3 & 8 & 2 & 1 \\ 0 & 0 & \textcircled{1} & 1 & 3 & \frac{1}{3} & 1 \\ 0 & 0 & 0 & 0 & 0 & \textcircled{1} & 1 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{3} \end{array}\right)$$

$$\xrightarrow{-3\mathcal{R}_4} \begin{pmatrix} \textcircled{1} & 2 & 2 & 3 & 8 & 2 & 1 \\ 0 & 0 & \textcircled{1} & 1 & 3 & \frac{1}{3} & 1 \\ 0 & 0 & 0 & 0 & 0 & \textcircled{1} & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \textcircled{1} \end{pmatrix} = A_9$$

Now we are ready to perform the Jordan part (Steps $6 \sim 8$).

$$A_9 \xrightarrow[-\mathcal{R}_4+\mathcal{R}_1]{\substack{-\mathcal{R}_4+\mathcal{R}_3 \\ -\mathcal{R}_4+\mathcal{R}_2}} \begin{pmatrix} \textcircled{1} & 2 & 2 & 3 & 8 & 2 & 0 \\ 0 & 0 & \textcircled{1} & 1 & 3 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & \textcircled{1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \textcircled{1} \end{pmatrix} \xrightarrow[-2\mathcal{R}_3+\mathcal{R}_1]{-\frac{1}{3}\mathcal{R}_3+\mathcal{R}_2} \begin{pmatrix} \textcircled{1} & 2 & 2 & 3 & 8 & 0 & 0 \\ 0 & 0 & \textcircled{1} & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \textcircled{1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \textcircled{1} \end{pmatrix}$$

$$\xrightarrow{-2\mathcal{R}_2+\mathcal{R}_1} \begin{pmatrix} \textcircled{1} & 2 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & \textcircled{1} & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \textcircled{1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \textcircled{1} \end{pmatrix} = A_{15}.$$

A_{15} is rref of A, denoted by $A \xrightarrow{rref} A_{15}$. ■

Example 3.2.8: Suppose $A = \begin{pmatrix} 1 & 1 & -4 & 1 & 3 \\ 2 & -3 & 7 & 7 & -4 \\ 0 & 1 & -3 & -1 & 2 \end{pmatrix} \xrightarrow{rref} H$. Find a matrix P such that $PA = H$.

We perform Guassian elimination to the following block matrix:

$$\left(\begin{array}{ccccc|ccc} 1 & 1 & -4 & 1 & 3 & 1 & 0 & 0 \\ 2 & -3 & 7 & 7 & -4 & 0 & 1 & 0 \\ 0 & 1 & -3 & -1 & 2 & 0 & 0 & 1 \end{array}\right)$$

■

Theorem 3.2.4: *Suppose $A \sim H$, where H is in rref, (i.e., $A \xrightarrow{rref} H$). Then H is unique.*

The proof will be provided later. By uniqueness, we use $\text{rref}(A)$ to denote such H in the above theorem.

Example 3.2.9: Solve the following system of linear equations over $\mathbb{R}$:

$$\begin{cases} x_1 + x_2 - 4x_3 + x_4 = 3 \\ 2x_1 - 3x_2 + 7x_3 + 7x_4 = -4 \\ x_2 - 3x_3 - x_4 = 2 \end{cases}$$

We first form the augmented matrix and perform elementary row operations to the matrix:

$$\left(\begin{array}{cccc|c} 1 & 1 & -4 & 1 & 3 \\ 2 & -3 & 7 & 7 & -4 \\ 0 & 1 & -3 & -1 & 2 \end{array}\right)$$

By Example 3.2.8 we have

$$\left(\begin{array}{cccc|c} 1 & 1 & -4 & 1 & 3 \\ 2 & -3 & 7 & 7 & -4 \\ 0 & 1 & -3 & -1 & 2 \end{array}\right) \xrightarrow{rref} \left(\begin{array}{cccc|c} 1 & 0 & -1 & 2 & 1 \\ 0 & 1 & -3 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

The equivalent system is

$$\begin{cases} x_1 - x_3 + 2x_4 = 1 \\ x_2 - 3x_3 - x_4 = 2 \end{cases} \tag{3.1}$$

While this system is fairly easy to solve, it also appears to have a multitude of solutions. For example, choose $x_3 = 1$, $x_4 = 0$ and see that then $x_1 = 2$ and $x_2 = 5$ will together form a solution. Or choose $x_3 = 0$, $x_4 = 1$, and then discover that $x_1 = -1$ and $x_2 = 3$ lead to a solution.

Pick *any* values of x_3 and x_4, we shall obtain solutions. Because of this behavior, we say that x_3 and x_4 are *free* or *independent* variables.

But why do we vary x_3 and x_4 and not some other variable(s)? For now, notice that the 3rd and the 4th columns of the augmented matrix is not a leading column. With this idea, we can rearrange the two equations, solving each for the variable whose index is the same as the column index of a leading column, i.e.,

$$\begin{aligned} x_1 &= 1 + x_3 - 2x_4 \\ x_2 &= 2 + 3x_3 + x_4 \end{aligned}$$

The set of solution is

$$\left\{ \begin{bmatrix} 1+a-2b \\ 2+3a+b \\ a \\ b \end{bmatrix} \middle| \; a, b \in \mathbb{R} \right\}.$$

Again, for saving space, we often write as

$$\{(1+a-2b,\ 2+3a+b,\ a,\ b) \mid a, b \in \mathbb{R}\}.$$

■

Question: Do we really find all solutions of the system? That is, is any solution not in this form?

We shall answer this question completely later after studying the structure of solution set of a linear system.

3.3 Type of Solution Sets

Definition 3.3.1: A system of linear equations is *consistent* if it has at least one solution. Otherwise, the system is called *inconsistent*.

Definition 3.3.2: Suppose A is the augmented matrix of a consistent system of linear equations and H is the row equivalent matrix in rref. Suppose j is the index of a leading column of H. Then the unknown x_j is called a *lead variable* (or *dependent variable*). An unknown that is not a lead variable is called a *free variable* (or *independent variable*).

Back to (3.1) of Example 3.2.9, x_1, x_2 are lead variables, and x_3, x_4 are free variables.
Here, you may see that lead variables are in terms of free variables.

Theorem 3.3.3: *Suppose $(A|\boldsymbol{b})$ is the augmented matrix of a system of linear equations, $\mathcal{LS}(A, \boldsymbol{b})$, with n unknowns. Suppose also that $(H|\boldsymbol{c})$ is a row equivalent matrix in reduced row echelon form with r nonzero rows.*

- *Then the system of equations is inconsistent if and only if column $n+1$ of $(H|\boldsymbol{c})$ is a leading column (i.e., the last column $\boldsymbol{c}$).*
- *Equivalently, the system is consistent if and only if $\boldsymbol{c}$ is not a leading column of $(H|\boldsymbol{c})$.*

Another way of expressing the theorem is to say that the system of equations is inconsistent if and only if the last non-zero row of $(H|\boldsymbol{c})$ is $(0, 0, \ldots, 0, 1)$.

Proof: Note that, it is easy to see that H is also in rref.

Example 3.3.1: Determine if the following system of linear equation is consistent.

$$\begin{cases} x_1 + x_2 + 2x_3 + 3x_4 + 2x_5 + 5x_6 = 1 \\ 2x_1 + 2x_2 + 3x_3 - x_4 = 1 \\ 3x_1 + 3x_2 + 5x_3 + x_4 + x_5 - 2x_6 = 3 \\ x_4 + x_5 + 7x_6 = 0 \end{cases}$$

Sometimes we do not write the 'brace' for the system. **Solution:**

Example 3.3.2: Determine whether the following system of linear equation is consistent.

$$\begin{array}{rcrcrcrcrcrcr} x_1 & + & x_2 & + & 2x_3 & + & 3x_4 & + & 2x_5 & + & 5x_6 & = & 1 \\ 2x_1 & + & 2x_2 & + & 3x_3 & - & x_4 & & & & & = & 1 \\ 3x_1 & + & 3x_2 & + & 5x_3 & + & x_4 & + & x_5 & - & 2x_6 & = & 3 \\ & & & & & & x_4 & + & x_5 & + & 7x_6 & = & -1 \end{array}$$

Solution:

Theorem 3.3.4: *Suppose $(A|\boldsymbol{b})$ is the augmented matrix of a consistent system of linear equations, $\mathcal{LS}(A, \boldsymbol{b})$, with n unknowns. Suppose also that $(H|\boldsymbol{c}) = \mathrm{rref}(A|\boldsymbol{b})$ with r pivots. Then $r \leq n$. If $r = n$, then the system has a unique solution; and if $r < n$, then the system has infinitely many solutions.*

Proof: Note that $(H|\boldsymbol{c})$ has $n+1$ columns. $(H|\boldsymbol{c})$ with r pivots implies that there are r leading columns.

3.4 Free variables

The next theorem simply states a conclusion from the final paragraph of the proof of Theorem 3.3.3, allowing us to state explicitly the number of free variables for a consistent system.

Theorem 3.4.1: *Suppose* $(A|\boldsymbol{b})$ *is the augmented matrix of a consistent system of linear equations with* n *unknowns. Suppose also that* $(H|\boldsymbol{c}) = \text{rref}(A|\boldsymbol{b})$ *with* r *pivots. Then the solution set can be described with* $n-r$ *free variables.*

Example 3.4.1:

1. System of linear equations with $n = 3$, $m = 3$.

$$\begin{array}{rcrcrcr} x_1 & - & x_2 & + & 2x_3 & = & 1 \\ 2x_1 & + & x_2 & + & x_3 & = & 8 \\ x_1 & + & x_2 & & & = & 5 \end{array}$$

Augmented matrix

$$\left(\begin{array}{rrr|r} 1 & -1 & 2 & 1 \\ 2 & 1 & 1 & 8 \\ 1 & 1 & 0 & 5 \end{array}\right)$$

The reduced row echelon form of the augmented matrix

$$\left(\begin{array}{ccr|c} \textcircled{1} & 0 & 1 & 3 \\ 0 & \textcircled{1} & -1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

The last column is not a leading column. So the system of linear equations is consistent.
$r = 2$, there is $3 - 2 = 1$ free variable.
In fact $D = \{1, 2\}$, $F = \{3\}$. x_1, x_2 are lead variables, x_3 is a free variable.

$$x_1 = 3 - x_3$$
$$x_2 = 2 + x_3$$

■

2. System of linear equations with $n = 3, m = 3$.

$$\begin{array}{rcrcrcr} -7x_1 & - & 6x_2 & - & 12x_3 & = & -33 \\ 5x_1 & + & 5x_2 & + & 7x_3 & = & 24 \\ x_1 & & & + & 4x_3 & = & 5 \end{array}$$

3. System of linear equations with $n = 2$, $m = 5$.

$$\begin{array}{rcrcr} 2x_1 & + & 3x_2 & = & 6 \\ -x_1 & + & 4x_2 & = & -14 \\ 3x_1 & + & 10x_2 & = & -2 \\ 3x_1 & - & x_2 & = & 20 \\ 6x_1 & + & 9x_2 & = & 18 \end{array}$$

4. System of linear equations with $n = 4, m = 3$.

$$\begin{array}{rcrcrcrcr} 2x_1 & + & x_2 & + & 7x_3 & - & 7x_4 & = & 2 \\ -3x_1 & + & 4x_2 & - & 5x_3 & - & 6x_4 & = & 3 \\ x_1 & + & x_2 & + & 4x_3 & - & 5x_4 & = & 2 \end{array}$$

Theorem 3.4.2: *A system of linear equations has no solutions, a unique solution or infinitely many solutions.*

Proof:

Theorem 3.4.3: *Suppose a consistent system of linear equations has m equations in n unknowns. If $n > m$, then the system has infinitely many solutions.*

Proof:

These theorems give us the procedures and implications that allow us to completely solve any system of linear equations. The main computational tool is using row operations to convert an augmented matrix into reduced row-echelon form. Here is an outline of how we would solve a system of linear equations.

Steps of solving system of linear equations

1. Represent a system of linear equations in n unknowns by an augmented matrix.

2. Convert the matrix to a row-equivalent matrix in rref using the Gaussian elimination. Identify the location of the leading columns, and the number of pivots r.

3. If column $n + 1$ is a leading column, then the system is inconsistent.

4. If column $n + 1$ is not a leading column, then there are two possibilities:

(a) $r = n$ and the solution is unique. It can be read off directly from the entries in rows 1 through n of column $n + 1$.

(b) $r < n$ and there are infinitely many solutions. We can describe the solution sets by the $n - r$ free variables.