Matrices

system of linear equations

$$\begin{aligned} 3x_1 + 2x_2 &= 5 \\ 2x_1 + x_2 &= 6 \end{aligned} \rightsquigarrow \underbrace{A = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}}_{\text{matrix of coefficients}}, \quad \underbrace{\vec{b} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}}_{\text{vector of constants}}$$

suppose $\boxed{(s_1, s_2)}$ is the unique solution to our system of equations. Write

$$\vec{x} = \begin{bmatrix} s_1 \\ s_2 \end{bmatrix}$$

for the "solution vector".

Ex: $3x_1 + 3x_2 = 2$ by elimination, we find that

$$6x_1 + 6x_2 = 4$$

the general solution is

$$\left\{ \left(a, \tfrac{2}{3} - a\right) \mid a \text{ is a real number} \right\}$$

$$A = \begin{bmatrix} 3 & 3 \\ 6 & 6 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} \quad \vec{x} = \begin{bmatrix} a \\ \tfrac{2}{3} - a \end{bmatrix}$$

a is a real number.

<u>Definition</u>: The <u>matrix representation</u> of a system of linear equations is the expression

$$A\vec{x} = \vec{b}$$

$\vec{b}$: vector of constants

A: matrix of coefficients

$\vec{x}$: ~~solution vector~~ vector of unknowns $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

Definition Augmented matrix

the augmented matrix associated to the system of equations $A\vec{x} = \vec{b}$ is the matrix

$$\text{m rows}\left\{\underbrace{\left[\begin{array}{c|c} \quad A \quad & \vec{b} \end{array}\right]}_{n+1 \text{ columns}}\right.$$

A $m \times n$ matrix

Ex: $A = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$ $\vec{b} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

$$A\vec{x} = \vec{b}$$

$$\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

augmented matrix: $\begin{array}{c} \quad A \qquad\quad \vec{b} \\ \left[\begin{array}{cc|c} 3 & 2 & 2 \\ 1 & 1 & 0 \end{array}\right] \end{array}$

what is this line? it has no mathematical meaning, it's just there to help us remember how we got our augmented matrix.

---

① $3x_1 + 5x_2 = 1$

② $3x_1 + 2x_2 = -2$

augmented matrix: $\left[\begin{array}{cc|c} 3 & 5 & 1 \\ 3 & 2 & -2 \end{array}\right]$

Solve:

① $3x_1 + \boxed{5x_2} = 1$

②' = ② − ① $0x_1 - 3x_2 = -3$

augmented matrix: $\left[\begin{array}{cc|c} 3 & 5 & 1 \\ 0 & -3 & -3 \end{array}\right]$

$$1' = 1 + \frac{5}{3} 2' \qquad 3x_1 + 0x_2 = -4 \qquad \left[\begin{array}{cc|c} 3 & 0 & -4 \\ 0 & -3 & -3 \end{array}\right]$$

$$2' \qquad 0 \cdot x_1 - 3x_2 = -3$$

$$1'' = \frac{1}{3} 1' \qquad x_1 + 0 \cdot x_2 = -4/3 \qquad \left[\begin{array}{cc|c} 1 & 0 & -4/3 \\ 0 & 1 & 1 \end{array}\right]$$

$$2'' = -\frac{1}{3} 2' \qquad 0 \cdot x_1 + x_2 = 1$$

Solve: $x_1 = -4/3$

$x_2 = 1$

solution vector $= \begin{bmatrix} -4/3 \\ 1 \end{bmatrix}$ = right-hand column of

---

Next goal: work only with augmented matrix

$[A \mid \vec{b}]$)

(stop using $x_1 \ldots x_n$, just use $[A \mid \vec{b}]$)

Ex: Suppose we want to solve

$$2x_1 + 5x_2 = 2$$
$$x_1 + 2x_2 = 0$$

$A = \begin{bmatrix} 2 & 5 \\ 1 & 2 \end{bmatrix}$ $\vec{b} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

augmented matrix: $\left[\begin{array}{cc|c} 2 & 5 & 2 \\ 1 & 2 & 0 \end{array}\right]$

$\rightsquigarrow$

solve!

$R_2' = R_2 - \frac{1}{2}R_1$ $\left[\begin{array}{cc|c} 2 & 5 & 2 \\ 0 & -\frac{1}{2} & -1 \end{array}\right]$

$\rightsquigarrow$

$$R_2'' = -2 \cdot R_2' \quad \left[\begin{array}{cc|c} 2 & 5 & 2 \\ 0 & 1 & 2 \end{array}\right]$$

$$\Downarrow$$

$$R_1' = R_1 - 5R_2'' \quad \left[\begin{array}{cc|c} 2 & 0 & -8 \\ 0 & 1 & 2 \end{array}\right]$$

$$\Downarrow$$

$$R_1'' = \tfrac{1}{2} R_1' \quad \left[\begin{array}{cc|c} 1 & 0 & -4 \\ 0 & 1 & 2 \end{array}\right]$$

vector of solutions to our original equation is

$$\boxed{\vec{x} = \begin{bmatrix} -4 \\ 2 \end{bmatrix}}$$

Check: $2x_1 + 5x_2 = 2$ $\qquad 2(-4) + 5(2) = 2$ ✓

$x_1 + 2x_2 = 0$ $\qquad (-4) + 2(2) = 0$ ✓

Row operations

(same thing as Equation operations but for augmented matrices)

Definition: a row operation is any one of the following 3 operations, which takes an augmented matrix $[A|\vec{b}]$ & produces a new augmented matrix $[A'|\vec{b}']$. (compare with equation operations)

1) Swap two rows. $R_i \leftrightarrow R_j$

Ex: $\left[\begin{array}{cc|c} 3 & 2 & 0 \\ 1 & 1 & 3 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 3 & 2 & 0 \end{array}\right]$

$R_1 \leftrightarrow R_2$

2) Multiply a row by a nonzero number. αR_i

α nonzero #

Ex: $\left[\begin{array}{cc|c} 3 & 2 & 0 \\ 1 & 1 & 3 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|c} 1 & 2/3 & 0 \\ 1 & 1 & 3 \end{array}\right]$

$\frac{1}{3}R_1$

3) Multiply row R_i by any number & add it to R_j $j \neq i$.

$R_j + \alpha R_i$

α a real number

Ex: $\left[\begin{array}{cc|c} 3 & 2 & 0 \\ 1 & 1 & 3 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|c} 0 & -1 & -9 \\ 1 & 1 & 3 \end{array}\right]$

$R_1 - 3R_2$