Chapter 2: Connectivity

2.1 Connectedness

Definition 2.1.1:

1. A *walk* W in a graph G is an alternating sequence $(u_0, e_1, u_1, e_2, \ldots, e_k, u_k)$ (or $u_0e_1u_1e_2 \cdots e_ku_k$, for short) of vertices and edges that begins and ends with a vertex, where $e_i = u_{i-1}u_i$ for each $i \in \{1, 2, \ldots, k\}$. The vertex u_0 is called the *initial vertex* of W and the vertex u_k is called the *final vertex* of W. The initial or final vertex of W is also called an *end vertex* of W and the natural number k is the *length* of W.

2. A *trail* in G is a walk with all of its edges distinct.

3. A *path* in G is a walk with all its vertices distinct.

4. A (u, v)*-walk* ((u, v)*-trail* or (u, v)*-path*) is a walk (respectively, trail or path) with initial vertex u and final vertex v.

5. A walk or trail of length at least one is *closed* if the initial vertex and the final vertex coincide. A close trail is also called a *circuit*.

6. A *cycle* is a closed walk with distinct vertices except the initial and final vertices coincide.

By convention, we consider a single vertex as a path (walk) of length zero. Such a path (walk) is called a *trivial path* (*walk*). However, cycles always have positive length and the only cycles of length 1 are loops. Also, the set of vertices and edges constitute a given walk, trail, path, or cycle in a graph G forms a subgraph of G.

If G is simple or there is no ambiguity about the edges being considered, then we simply write a walk, trial, path, or cycle by a sequence of vertices $u_0u_1 \cdots u_k$ instead of $(u_0, e_1, u_1, e_2, \ldots, e_k, u_k)$.

Definition 2.1.2: Let $P = (u_0e_1u_1e_2 \cdots e_ku_k)$ and $Q = (v_0f_1v_1f_2 \cdots f_kv_l)$ be two walks in a graph. If $u_k = v_0$, then the *composite* walk is formed by

$$PQ = (u_0e_1u_1e_2 \cdots e_ku_kf_1v_1f_2 \cdots f_kv_l).$$

The *inverse* walk of P is defined by $P^{-1} = (u_ke_k \cdots u_1e_1u_0)$.

Lemma 2.1.3: *Let G be a graph having distinct vertices u and v. Any (u, v)-walk contains a (u, v)-path.*

Corollary 2.1.4: *Suppose W is a circuit. For any $u \in V(W)$, there is a cycle in W containing u.*

Definition 2.1.5: Two vertices u and v are *connected* in a graph G if there is a (u, v)-path in G. A graph G is *connected* if every pair of distinct vertices $u, v \in V(G)$ are connected. Otherwise G is *disconnected*.

By Lemma 2.1.3, the term (u, v)-path in the above definition can be replaced by (u, v)-walk.

Proposition 2.1.6: *Let $G = (V, E)$ be a graph. Connectivity on V is an equivalence relation**.

*Readers can refer to any algebra textbook for the formal definition of equivalence relation

Let $G = (V, E)$ and let $V_1, \dots, V_\omega$ be equivalence classes of the equivalence relation $\sim$. Then $G[V_1], \dots, G[V_\omega]$ are pairwise disjoint subgraphs and $G = G[V_1] + \cdots + G[V_\omega] = \sum_{i=1}^{\omega} G[V_i]$.

Definition 2.1.7: Undertake the defined symbols above, $G[V_1], \dots, G[V_\omega]$ are called *connected components* (or simply *components*) of G. We use $\omega(G)$ to denote the number of component(s) of G.

That is, G is connected if and only if $\omega(G) = 1$.

Theorem 2.1.8: *If a graph has exactly two vertices of odd degree, then these two vertices must be connected.*

Theorem 2.1.9: *For a simple graph G of order p and ω components, we have*

$$|E(G)| \le \frac{(p-\omega)(p-\omega+1)}{2}.$$

Definition 2.1.10: A vertex v is a *cut-vertex* if $G - v$ has more components than G. An edge e is a *cut-edge* (or *bridge*) if $G - e$ has more components than G.

Lemma 2.1.11: *The number of components $\omega(G) \le \omega(G - e) \le \omega(G) + 1$ for any edge e of G.*

Corollary 2.1.12: *For a graph G and $e \in E(G)$, the following are equivalent:*

(1) The edge e is a bridge of G.

(2) The edge e is not contained in any cycle of G.

2.2 Distance

Definition 2.2.1: Let $G = (V, E)$ be a graph. For $u, v \in V$, the *distance* between u and v, denoted $d_G(u, v)$ (or $d(u, v)$ when there is no ambiguity), is the length of the shortest (u, v)-path in G. If there is no path between them in G, then we assign $d_G(u, v) = \infty$.

Note that $d_G(\cdot, \cdot)$ is a metric on G. And if $H \subseteq G$, then $d_G(u, v) \le d_H(u, v)$ for all $u, v \in V(H)$.

Definition 2.2.2: Let G be a graph and $u \in V(G)$.

1. The *eccentricity* $\epsilon_G(u)$ (or $\epsilon(u)$) of u in G is the distance from u to the vertex farthest from u in G. That is,

$$\epsilon_G(u) = \max_{v \in V(G)} \{d_G(u, v)\}.$$

2. A *center* of G is a vertex having minimum eccentricity.

3. The eccentricity of a center of G is called the *radius* of G and denoted by $\mathrm{rad}(G)$.

4. The *diameter* of G is defined by

$$\mathrm{diam}(G) = \max_{u,v \in V(G)} \{d_G(u, v)\} = \max_{u \in V(G)} \{\epsilon_G(u)\}.$$

Suppose G is a graph with diameter k. Then there are two vertices u and v such that $d(u, v) = k$, which implies there is a (u, v)-path P of length k. Such a path is called a *diameter* (or *diametral path*) of G.

2.3 Edge Cuts

Definition 2.3.1: Let G be a connected graph. An *edge cut* S is a set of edges such that the graph $G - S$ is disconnected and $G - S'$ is connected for any subset $S' \subset S$.

Remark 2.3.2: The above definition differs from some books. In most books, it is called a *bond*, and "edge cut" has another meaning. Note that an edge cut is a minimal set of edges that disconnects a connected graph. A graph may contains many edge cuts. Recalled that if $S = \{e\}$ is an edge cut, then e is a cut-edge (bridge).

Lemma 2.3.3: *If S is an edge cut of a connected graph G, then $G - S$ has precisely two components.*

Theorem 2.3.4: *Let G be a connected graph. If C is a cycle in G and S is an edge cut of G, then $|E(C) \cap S|$ is even.*

2.4 Edge Connectivity and Connectivity

Definition 2.4.1: Let G be a graph with two or more vertices. The smallest cardinal of an edge cut S of G is called the *edge-connectivity* of G, denoted by $\kappa'(G)$ (or κ'). If $k \leq \kappa'(G)$, then we say that G is *k-edge-connected*.

Remark 2.4.2: Note that

1. For any disconnected graph G, we have $\kappa'(G) = 0$.
2. A connected graph G has a bridge if and only if $\kappa'(G) = 1$.
3. If G is a graph and G' is the graph obtained from G by removing all of its loops, then $\kappa'(G) = \kappa'(G')$.
4. If $G = N_1$, then we define $\kappa'(G) = \infty$ by convention.

Similar to edge-connectivity we may define vertex-connectivity.

Definition 2.4.3: Let G be a graph. The minimum number of vertices of G, whose removal disconnects G or creates a graph with a single vertex, is called the *connectivity* of G and is denoted by $\kappa(G)$ (or κ). If $k \leq \kappa(G)$, then we say that G is *k-connected*.

Remark 2.4.4: Note that

1. For any disconnected graph G, we have $\kappa(G) = 0$.
2. If G is a graph and G' is the graph obtained from G by removing all of its loops and collapsing all multiple edges to single edges, then $\kappa(G) = \kappa(G')$.

Example 2.4.5: For $m, n \geq 2$ and $p \geq 1$,

1. $\kappa(N_p) = 0$.
2. $\kappa(C_p) = 2$ if $p \geq 3$.
3. $\kappa(K_p) = p - 1$.
4. $\kappa(K_{m,n}) = \min\{m, n\}$.

Theorem 2.4.6: *For a graph G, we have $\kappa(G) \le \kappa'(G) \le \delta(G)$.*

Definition 2.4.8: Let G be a graph and $u, v \in V(G)$. If u and v are the only common vertices of two (u,v)-paths P and Q, then these two paths are called *internally disjoint*.

Theorem 2.4.9 (Whitney, 1932)**:** *For a connected graph $G = (V,E)$ has three or more vertices, the following statements are equivalent:*

(1) G is 2-connected.

(2) For every pair of distinct vertices, there is a cycle in G contains both of them.

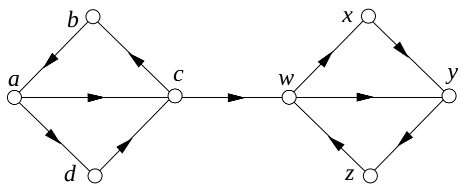
(3) For every pair of distinct vertices, there are two internally disjoint paths in G connecting them.

2.5 Connectedness for Digraphs

In digraph, the concept of connectedness is slightly different from undirected graph.

Definition 2.5.1: A digraph is *connected* (or *weakly connected*) if its underlying graph is connected. A *component* of $\overrightarrow{G}$ means the subdigraph induced by the vertices of the corresponding component of the underlying graph G.

Example 2.5.2: Consider the following digraph.



Clearly it is connected, but it is easy to see that we cannot travel c from w in the above digraph.

Hence the connectivity defined above seems to be different with our intuition. Therefore, we introduce the following definitions.

Definition 2.5.3: A *directed walk* $\overrightarrow{W}$ in a digraph $\overrightarrow{G}$ is an alternating sequence

$$\overrightarrow{W} = (u_0, e_1, u_1, e_2, \ldots, e_k, u_k) \text{ (or } \overrightarrow{W} = u_0e_1u_1e_2\cdots e_ku_k, \text{ for short)}$$

of vertices and arcs. Where, for each $i \in \{1, 2, \ldots, k\}$, the tail and head of e_i are u_{i-1} and u_i, respectively.

The definitions of directed trail, path, cycle, etc. are similar to undirected graph and we omit the details here.

Remark 2.5.4: Since each arc has a unique tail and head, there is no ambiguity in writing a directed walk as $\overrightarrow{W} = e_1e_2\cdots e_k$, where it is understood that the initial vertex of $\overrightarrow{W}$ is the tail of e_1 and the final vertex is the head of e_k.

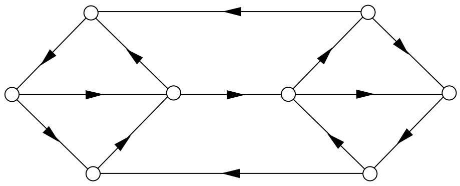
Likewise, if our digraph has no parallel arcs, then we can write a directed walk as a sequence of vertices $\overrightarrow{W} = u_0u_1\cdots u_k$.

Lemma 2.5.5: *Let $\overrightarrow{G}$ be a digraph having distinct vertices u and v. Any directed (u,v)-walk contains a directed (u,v)-path.*

Definition 2.5.6: Let $\overrightarrow{G} = (V, E)$ be a digraph and let $u, v \in V$. If there is a directed (u, v)-path, then we say that u *is reachable to* v (or v *is reachable from* u).

Definition 2.5.7: Let $\overrightarrow{G} = (V, E)$ be a digraph. If every vertex is reachable to others, then $\overrightarrow{G}$ is called *strongly connected*. The *strong component* of $\overrightarrow{G}$ is a maximal strongly connected subdigraph of $\overrightarrow{G}$.

Example 2.5.8: The following is a strongly connected digraph.



Definition 2.5.9: Let G be an undirected graph. An *orientation* of G is a digraph obtained from G by assigning each edge a direction. If there is a strongly connected orientation of G, then we call G *orientable*.

Theorem 2.5.10 (Robbins, 1939)**:** *A connected graph $G = (V, E)$ is orientable if and only if it has no bridges.*