Chapter 2: Connectivity

2.1 Connectedness

Definition 2.1.1:

1. A walk $W$ in a graph $G$ is an alternating sequence $(u_0, e_1, u_1, e_2, \ldots, e_k, u_k)$ (or $u_0e_1u_1e_2\cdots e_ku_k$, for short) of vertices and edges that begins and ends with a vertex, where $e_i = u_{i-1}u_i$ for each $i \in \{1, 2, \ldots, k\}$. The vertex $u_0$ is called the initial vertex of $W$ and the vertex $u_k$ is called the final vertex of $W$. The initial or final vertex of $W$ is also called an end vertex of $W$ and the natural number $k$ is the length of $W$.

2. A trail in $G$ is a walk with all of its edges distinct.

3. A path in $G$ is a walk with all its vertices distinct.

4. A $(u, v)$-walk ($(u, v)$-trail or $(u, v)$-path) is a walk (respectively, trail or path) with initial vertex $u$ and final vertex $v$.

5. A walk or trail of length at least one is closed if the initial vertex and the final vertex coincide. A closed trail is also called a circuit.

6. A cycle is a closed walk with distinct vertices except the initial and final vertices coincide.

By convention, we consider a single vertex as a path (walk) of length zero. Such a path (walk) is called a trivial path (walk). However, cycles always have positive length and the only cycles of length 1 are loops. Also, the set of vertices and edges constitute a given walk, trail, path, or cycle in a graph $G$ forms a subgraph of $G$.

If $G$ is simple or there is no ambiguity about the edges being considered, then we simply write a walk, trial, path, or cycle by a sequence of vertices $u_0u_1\cdots u_k$ instead of $(u_0, e_1, u_1, e_2, \ldots, e_k, u_k)$.

Definition 2.1.2: Let $P = (u_0e_1u_1e_2\cdots e_ku_k)$ and $Q = (v_0f_1v_1f_2\cdots f_lv_l)$ be two walks in a graph. If $u_k = v_0$, then the composite walk is formed by

$$PQ = (u_0e_1u_1e_2\cdots e_ku_kf_1v_1f_2\cdots f_lv_l).$$

The inverse walk of $P$ is defined by $P^{-1} = (u_k\cdots u_1e_1u_0)$.

Lemma 2.1.3: Let $G$ be a graph having distinct vertices $u$ and $v$. Any $(u, v)$-walk contains a $(u, v)$-path.

Corollary 2.1.4: Suppose $W$ is a circuit. For any $u \in V(W)$, there is a cycle in $W$ containing $u$.

Definition 2.1.5: Two vertices $u$ and $v$ are connected in a graph $G$ if there is a $(u, v)$-path in $G$. A graph $G$ is connected if every pair of distinct vertices $u, v \in V(G)$ are connected. Otherwise $G$ is disconnected.

By Lemma 2.1.3, the term $(u, v)$-path in the above definition can be replaced by $(u, v)$-walk.

Proposition 2.1.6: Let $G = (V, E)$ be a graph. Connectivity on $V$ is an equivalence relation$^*$. 

$^*$Readers can refer to any algebra textbook for the formal definition of equivalence relation.
Let $G = (V,E)$ and let $V_1, \ldots, V_\omega$ be equivalence classes of the equivalence relation $\sim$. Then $G[V_1], \ldots, G[V_\omega]$ are pairwise disjoint subgraphs and $G = G[V_1] + \cdots + G[V_\omega] = \sum_{i=1}^\omega G[V_i]$.

**Definition 2.1.7:** Undertake the defined symbols above, $G[V_1], \ldots, G[V_\omega]$ are called connected components (or simply components) of $G$. We use $\omega(G)$ to denote the number of component(s) of $G$.

That is, $G$ is connected if and only if $\omega(G) = 1$.

**Theorem 2.1.8:** If a graph has exactly two vertices of odd degree, then these two vertices must be connected.

**Theorem 2.1.9:** For a simple graph $G$ of order $p$ and $\omega$ components, we have

$$|E(G)| \leq \frac{(p-\omega)(p-\omega+1)}{2}.$$

**Definition 2.1.10:** A vertex $v$ is a cut-vertex if $G - v$ has more components than $G$. An edge $e$ is a cut-edge (or bridge) if $G - e$ has more components than $G$.

**Lemma 2.1.11:** The number of components $\omega(G) \leq \omega(G - e) \leq \omega(G) + 1$ for any edge $e$ of $G$.

**Corollary 2.1.12:** For a graph $G$ and $e \in E(G)$, the following are equivalent:

1. The edge $e$ is a bridge of $G$.
2. The edge $e$ is not contained in any cycle of $G$.

### 2.2 Distance

**Definition 2.2.1:** Let $G = (V,E)$ be a graph. For $u, v \in V$, the distance between $u$ and $v$, denoted $d_G(u,v)$ (or $d(u,v)$ when there is no ambiguity), is the length of the shortest $(u,v)$-path in $G$. If there is no path between them in $G$, then we assign $d_G(u,v) = \infty$.

Note that $d_G(\cdot, \cdot)$ is a metric on $G$. And if $H \subseteq G$, then $d_G(u,v) \leq d_H(u,v)$ for all $u,v \in V(H)$.

**Definition 2.2.2:** Let $G$ be a graph and $u \in V(G)$.

1. The eccentricity $\epsilon_G(u)$ (or $\epsilon(u)$) of $u$ in $G$ is the distance from $u$ to the vertex farthest from $u$ in $G$. That is,

$$\epsilon_G(u) = \max_{v \in V(G)} \{d_G(u,v)\}.$$

2. A center of $G$ is a vertex having minimum eccentricity.

3. The eccentricity of a center of $G$ is called the radius of $G$ and denoted by rad($G$).

4. The diameter of $G$ is defined by

$$\text{diam}(G) = \max_{u,v \in V(G)} \{d_G(u,v)\} = \max_{u \in V(G)} \{\epsilon_G(u)\}.$$

Suppose $G$ is a graph with diameter $k$. Then there are two vertices $u$ and $v$ such that $d(u,v) = k$, which implies there is a $(u,v)$-path $P$ of length $k$. Such a path is called a diameter (or diametral path) of $G$. 

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2.3 Edge Cuts

**Definition 2.3.1:** Let $G$ be a connected graph. An *edge cut* $S$ is a set of edges such that the graph $G - S$ is disconnected and $G - S'$ is connected for any subset $S' \subset S$.

**Remark 2.3.2:** The above definition differs from some books. In most books, it is called a *bond*, and “edge cut” has another meaning. Note that an edge cut is a minimal set of edges that disconnects a connected graph. A graph may contain many edge cuts. Recalled that if $S = \{e\}$ is an edge cut, then $e$ is a cut-edge (bridge).

**Lemma 2.3.3:** If $S$ is an edge cut of a connected graph $G$, then $G - S$ has precisely two components.

**Theorem 2.3.4:** Let $G$ be a connected graph. If $C$ is a cycle in $G$ and $S$ is an edge cut of $G$, then $|E(C) \cap S|$ is even.

2.4 Edge Connectivity and Connectivity

**Definition 2.4.1:** Let $G$ be a graph with two or more vertices. The smallest cardinal of an edge cut $S$ of $G$ is called the *edge-connectivity* of $G$, denoted by $\kappa'(G)$ (or $\kappa'$). If $k \leq \kappa'(G)$, then we say that $G$ is $k$-edge-connected.

**Remark 2.4.2:** Note that
1. For any disconnected graph $G$, we have $\kappa'(G) = 0$.
2. A connected graph $G$ has a bridge if and only if $\kappa'(G) = 1$.
3. If $G$ is a graph and $G'$ is the graph obtained from $G$ by removing all of its loops, then $\kappa'(G) = \kappa'(G')$.
4. If $G = N_1$, then we define $\kappa'(G) = \infty$ by convention.

Similar to edge-connectivity we may define vertex-connectivity.

**Definition 2.4.3:** Let $G$ be a graph. The minimum number of vertices of $G$, whose removal disconnects $G$ or creates a graph with a single vertex, is called the *connectivity* of $G$ and is denoted by $\kappa(G)$ (or $\kappa$). If $k \leq \kappa(G)$, then we say that $G$ is $k$-connected.

**Remark 2.4.4:** Note that
1. For any disconnected graph $G$, we have $\kappa(G) = 0$.
2. If $G$ is a graph and $G'$ is the graph obtained from $G$ by removing all of its loops and collapsing all multiple edges to single edges, then $\kappa(G) = \kappa(G')$.

**Example 2.4.5:** For $m, n \geq 2$ and $p \geq 1$,
1. $\kappa(N_p) = 0$.
2. $\kappa(C_p) = 2$ if $p \geq 3$.
3. $\kappa(K_p) = p - 1$.
4. $\kappa(K_{m,n}) = \min\{m, n\}$. 

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Theorem 2.4.6: For a graph $G$, we have $\kappa(G) \leq \kappa'(G) \leq \delta(G)$.

Definition 2.4.8: Let $G$ be a graph and $u, v \in V(G)$. If $u$ and $v$ are the only common vertices of two $(u,v)$-paths $P$ and $Q$, then these two paths are called internally disjoint.

Theorem 2.4.9 (Whitney, 1932): For a connected graph $G = (V, E)$ has three or more vertices, the following statements are equivalent:

1. $G$ is 2-connected.
2. For every pair of distinct vertices, there is a cycle in $G$ contains both of them.
3. For every pair of distinct vertices, there are two internally disjoint paths in $G$ connecting them.

2.5 Connectedness for Digraphs

In digraph, the concept of connectedness is slightly different from undirected graph.

Definition 2.5.1: A digraph is connected (or weakly connected) if its underlying graph is connected. A component of $\overrightarrow{G}$ means the subdigraph induced by the vertices of the corresponding component of the underlying graph $G$.

Example 2.5.2: Consider the following digraph.

![Digraph Example](image)

Clearly it is connected, but it is easy to see that we cannot travel $c$ from $w$ in the above digraph.

Hence the connectivity defined above seems to be different with our intuition. Therefore, we introduce the following definitions.

Definition 2.5.3: A directed walk $\overrightarrow{W}$ in a digraph $\overrightarrow{G}$ is an alternating sequence $\overrightarrow{W} = (u_0, e_1, u_1, e_2, \ldots, e_k, u_k)$ (or $\overrightarrow{W} = u_0 e_1 u_1 e_2 \cdots e_k u_k$, for short) of vertices and arcs. Where, for each $i \in \{1, 2, \ldots, k\}$, the tail and head of $e_i$ are $u_{i-1}$ and $u_i$, respectively.

The definitions of directed trail, path, cycle, etc. are similar to undirected graph and we omit the details here.

Remark 2.5.4: Since each arc has a unique tail and head, there is no ambiguity in writing a directed walk as $\overrightarrow{W} = e_1 e_2 \cdots e_k$, where it is understood that the initial vertex of $\overrightarrow{W}$ is the tail of $e_1$ and the final vertex is the head of $e_k$.

Likewise, if our digraph has no parallel arcs, then we can write a directed walk as a sequence of vertices $\overrightarrow{W} = u_0 u_1 \cdots u_k$.

Lemma 2.5.5: Let $\overrightarrow{G}$ be a digraph having distinct vertices $u$ and $v$. Any directed $(u,v)$-walk contains a directed $(u,v)$-path.
Definition 2.5.6: Let $\overrightarrow{G} = (V, E)$ be a digraph and let $u, v \in V$. If there is a directed $(u, v)$-path, then we say that $u$ is reachable to $v$ (or $v$ is reachable from $u$).

Definition 2.5.7: Let $\overrightarrow{G} = (V, E)$ be a digraph. If every vertex is reachable to others, then $\overrightarrow{G}$ is called strongly connected. The strong component of $\overrightarrow{G}$ is a maximal strongly connected subdigraph of $\overrightarrow{G}$.

Example 2.5.8: The following is a strongly connected digraph.

\[ \begin{array}{c}
\text{Diagram}
\end{array} \]

Definition 2.5.9: Let $G$ be an undirected graph. An orientation of $G$ is a digraph obtained from $G$ by assigning each edge a direction. If there is a strongly connected orientation of $G$, then we call $G$ orientable.

Theorem 2.5.10 (Robbins, 1939): A connected graph $G = (V, E)$ is orientable if and only if it has no bridges.