THE CHINESE UNIVERSITY OF HONG KONG
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MMAT5360 Game Theory

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1 Two-person zero sum games

1.1 Game matrices

In a two-person zero sum game, two players, player I and player II, make their moves simultaneously. Then the payoffs to the players depend on the strategies used by the players. In this chapter, we study only **zero sum games** which means the sum of the payoffs to the players is always zero. We will also assume that the game has **perfect information** which means all players know how the outcomes depend on the strategies the players use.

Definition 1.1.1 (Strategic form of a two-person zero sum game)**.** *The* **strategic form** *of a two-person zero sum game is given by a triple* (X, Y, π) *where*

1. *X is the set of strategies of player I.*

2. *Y is the set of strategies of player II.*

3. *$\pi : X \times Y \to \mathbb{R}$ is the payoff function of player I.*

For $(x, y) \in X \times Y$, the value $\pi(x, y)$ is the payoff to player I when player I uses strategy x and player II uses strategy y. Note that the payoff to player II is equal to $-\pi(x, y)$ since the game is a zero sum game. The game has perfect information means that the function π is known to both players. We will always assume that the sets X and Y are finite. In this case we may assume $X = \{1, 2, \cdots, m\}$ and $Y = \{1, 2, \cdots, n\}$. Then the payoff function can be represented by an $m \times n$ matrix which is called the **game matrix** and we will denote it by $A = [a_{ij}]$. A two-person zero sum game is completely determined by its game matrix. When player I uses the i-th strategy and player II uses the j-th strategy, then the payoff to player I is the entry a_{ij} of A. The payoff to player II is then $-a_{ij}$. If a two-person zero sum game is represented by a game matrix, we will call player I the **row player** and player II the **column player**.

Given a game matrix A, we would like to know what the optimal strategies for the players are and what the payoffs to the players will be if both of them use their optimal strategies. The answer to this question is simple if A has a saddle point.

Definition 1.1.2 (Saddle point)**.** *We say that an entry a_{kl} is a* **saddle point** *of an $m \times n$ matrix A if*

1. $a_{kl} = \min_{j=1,2,\cdots,n}\{a_{kj}\}$

2. $a_{kl} = \max_{i=1,2,\cdots,m}\{a_{il}\}$

The first condition means that when the row player uses the k-th strategy, then the payoff to the row player is not less than a_{kl} no matter how the column player plays. The second condition means that when the column player uses the l-th strategy, then the payoff to the row player is not larger than a_{kl} no matter how the row player plays. Consequently we have

Theorem 1.1.3. *If A has a saddle point a_{kl}, then the row player may guarantee that his payoff is not less than a_{kl} by using the k-th strategy and the column player may guarantee that the payoff to the row player is not larger than a_{kl} by using the l-th strategy.*

Suppose A is a matrix which has a saddle point a_{kl}. The above theorem implies that the corresponding row and column constitute the optimal strategies for the players. To find the saddle points of a matrix, first write down the row minima of the rows and the column maxima of the columns. Then find the maximum of row minima which is called the **maximin**, and the minimum of the column maxima which is called the **minimax**. If the maximin is equal to the minimax, then the entry in the corresponding row and column is a saddle point. If the maximin and minimax are different, then the matrix has no saddle point.

Example 1.1.4.

$$\begin{array}{cc} & \begin{array}{c} \\ \end{array} \\ & \begin{pmatrix} 1 & 2 & 0 \\ 3 & 5 & 2 \\ 0 & -4 & -3 \\ -2 & 4 & 1 \end{pmatrix} \\ \textit{max} & \begin{array}{ccc} 3 & 5 & 2 \end{array} \end{array} \begin{array}{c} \textit{min} \\ 0 \\ 2 \\ -4 \\ -2 \\ \\ \end{array}$$

Both the maximin and minimax are 2. Therefore the entry $a_{23} = 2$ is a saddle point. □

Example 1.1.5.

$$\begin{array}{cc} & \begin{array}{c} min \end{array} \\ \begin{pmatrix} 2 & -1 & 3 & 1 \\ -4 & 2 & 0 & 3 \\ 0 & 1 & -2 & 4 \end{pmatrix} & \begin{array}{c} -1 \\ -4 \\ -2 \end{array} \\ max \quad \begin{array}{cccc} 2 & 2 & 3 & 4 \end{array} & \end{array}$$

The maximin is -1 *while the minimax is* 2 *which are not equal. Therefore the matrix has no saddle point.* □

Saddle point of a matrix may not be unique. However the values of saddle points are always the same.

Theorem 1.1.6. *The values of the saddle points of a matrix are the same. That is to say, if* a_{kl} *and* a_{pq} *are saddle points of a matrix, then* $a_{kl} = a_{pq}$*. Furthermore, we have* $a_{pq} = a_{pl} = a_{kq} = a_{kl}$*.*

Proof. We have

$$\begin{aligned} a_{kl} &\leq a_{kq} \quad (\text{since } a_{kl} \leq a_{kj} \text{ for any } j) \\ &\leq a_{pq} \quad (\text{since } a_{iq} \leq a_{pq} \text{ for any } i) \\ &\leq a_{pl} \quad (\text{since } a_{pq} \leq a_{pj} \text{ for any } j) \\ &\leq a_{kl} \quad (\text{since } a_{il} \leq a_{kl} \text{ for any } i) \end{aligned}$$

Therefore

$$a_{kl} = a_{kq} = a_{pq} = a_{pl}$$

□

We have seen that if A has a saddle point, then the two players have optimal strategies by using one of their strategies constantly (Theorem 1.1.3). If A has no saddle point, it is expected that the optimal ways for the players to play the game are not using one of the strategies constantly. Take the rock-paper-scissors game as an example.

Example 1.1.7 (Rock-paper-scissors)**.** *The* **rock-paper-scissors** *game has the game matrix*

$$\begin{array}{cc} & \begin{array}{ccc} R & P & S \end{array} \\ \begin{array}{c} R \\ P \\ S \end{array} & \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix} \end{array}$$

Here we use the order rock(R), paper(P), scissors(S) to write down the game matrix. □

Everybody knows that the optimal strategy of playing the rock-paper-scissors game is not using any one of the gestures constantly. When one of the strategies of a player is used constantly, we say that it is a **pure strategy**. For games without saddle point like rock-paper-scissors game, mixed strategies instead of pure strategies should be used.

Definition 1.1.8 (Mixed strategy)**.** *A* **mixed strategy** *is a row vector* $\mathbf{x} = (x_1, x_2, \cdots, x_m) \in \mathbb{R}^m$ *such that*

1. $x_i \geq 0$ *for any* $i = 1, 2, \cdots, m$
2. $\sum_{i=1}^{m} x_i = 1$

In other words, a vector is a mixed strategy if it is a **probability vector***. We will denote the set of probability* m *vectors by* $\mathcal{P}^m$*.*

When a mixed strategy $(x_1, x_2, \cdots, x_m)$ is used, the player uses his i-th strategy with a probability of x_i for $i = 1, 2, \cdots, m$. Mixed strategies are generalization of pure strategies. If one of the coordinates of a mixed strategy is 1 and all other coordinates are 0, then it is a pure strategy. So a pure strategy is also a mixed strategy. Suppose the row player and the column player use mixed strategies $\mathbf{x} \in \mathcal{P}^m$ and $\mathbf{y} \in \mathcal{P}^n$ respectively. Then the outcome of the game is not fixed because the payoffs to the players will then be random variables. We denote by $\pi(\mathbf{x}, \mathbf{y})$ the **expected payoff** to the row player when the row player uses mixed strategy $\mathbf{x}$ and the column player uses mixed strategy $\mathbf{y}$. We have the following simple formula for the expected payoff $\pi(\mathbf{x}, \mathbf{y})$ to the row player.

Theorem 1.1.9. *In a two-person zero sum game with* $m \times n$ *game matrix* A*, suppose the row player uses mixed strategies* $\mathbf{x}$ *and the column player uses mixed strategies* $\mathbf{y}$ *independently. Then the expected payoff to the row player is*

$$\pi(\mathbf{x}, \mathbf{y}) = \mathbf{x} A \mathbf{y}^T$$

where $\mathbf{y}^T$ *is the transpose of* $\mathbf{y}$*.*

Proof. The expected payoff to the row player is

$$
\begin{aligned}
& E(\text{payoff to the row player}) \\
= & \sum_{\substack{1 \le i \le m \\ 1 \le j \le n}} a_{ij} P(I \text{ uses } i\text{-th strategy and } II \text{ uses } j\text{-th strategy}) \\
= & \sum_{\substack{1 \le i \le m \\ 1 \le j \le n}} a_{ij} P(I \text{ uses } i\text{-th strategy}) P(II \text{ uses } j\text{-th strategy}) \\
= & \sum_{\substack{1 \le i \le m \\ 1 \le j \le n}} a_{ij} x_i y_j \\
= & \mathbf{x} A \mathbf{y}^T
\end{aligned}
$$

□

Let A be an $m \times n$ game matrix. For $\mathbf{x} \in \mathcal{P}^m$, the vector

$$\mathbf{x}A \in \mathbb{R}^n$$

has the following interpretation. The j-th coordinate, $j = 1, 2, \cdots, n$, of the vector is the expected payoff to the row player if the row player uses mixed strategy $\mathbf{x}$ and the column player uses the j-th strategy constantly. In this case a rational column player would use the l-th strategy, $1 \le l \le n$, such that the l-th coordinate of the vector $\mathbf{x}A$ is the least coordinate among all coordinates of $\mathbf{x}A$. (Note that the column player wants the expected payoff to the row player as small as possible since the game is a zero sum game.)

On the other hand, for $\mathbf{y} \in \mathcal{P}^n$, the i-th coordinate, $i = 1, 2, \cdots, m$, of the column vector

$$A\mathbf{y}^T \in \mathbb{R}^m$$

is the expected payoff to the row player if the row player uses his i-th strategy constantly and the column player uses the mixed strategy $\mathbf{y}$. In this case a rational row player would use the k-th strategy, $1 \le k \le m$, such that the k-th coordinate of $A\mathbf{y}^T$ is the largest coordinate among all coordinates of $A\mathbf{y}^T$.

When a game matrix does not have a saddle point, both players do not have optimal pure strategies. However there always exists optimal mixed strategies for the players by the following minimax theorem due to von Neumann.

Theorem 1.1.10 (Minimax theorem)**.** *Let A be an $m \times n$ matrix. Then there exists real number $\nu \in \mathbb{R}$, mixed strategy for the row player $\mathbf{p} \in \mathbb{R}^m$, and mixed strategy for the column player $\mathbf{q} \in \mathbb{R}^n$ such that*

1. $\mathbf{p}A\mathbf{y}^T \geq \nu$, *for any* $\mathbf{y} \in \mathcal{P}^n$

2. $\mathbf{x}A\mathbf{q}^T \leq \nu$, *for any* $\mathbf{x} \in \mathcal{P}^m$

3. $\mathbf{p}A\mathbf{q}^T = \nu$

In the above theorem, the real number $\nu = \nu(A)$ is called the **value**, or the **security level**, of the game matrix A. The strategy $\mathbf{p}$ is called a **maximin strategy** for the row player and the strategy $\mathbf{q}$ is called a **minimax strategy** for the column player. The value ν of a matrix is unique. However maximin strategy and minimax strategy are in general not unique.

The maximin strategy $\mathbf{p}$ and the minimax strategy $\mathbf{q}$ are the optimal strategies for the row player and the column player respectively. It is because the row player may guarantee that his payoff is at least ν no matter how the column player plays by using the maximin strategy $\mathbf{p}$. This is also the reason why the value ν is called the security level. Similarly, the column player may guarantee that the payoff to the row player is at most ν, and thus his payoff is at least $-\nu$, no matter how the row player plays by using the minimax strategy $\mathbf{q}$. We will prove the minimax theorem in Section 2.4.

1.2 2×2 games

In this section, we study 2×2 game matrices closely and see how one can solve them, that means finding the maximin strategies for the row player, minimax strategies for the column player and the values of the game. First we look at a simple example.

Example 1.2.1 (Modified rock-paper-scissors)**.** *The rules of the modified rock-paper-scissors are the same as the ordinary rock-paper-scissors except that the row player can only show the gesture rock(R) or paper(P) but not scissors while the column player can only show the gesture scissors(S) or rock(R) but not paper. The game matrix of the game is*

$$\begin{array}{cc} & \begin{array}{cc} S & R \end{array} \\ \begin{array}{c} R \\ P \end{array} & \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \end{array}$$

It is easy to see that the game matrix does not have a saddle point. Thus there is no pure maximin or minimax strategy. To solve the game, suppose *the row player uses mixed strategy* $\mathbf{x} = (x, 1-x)$. *Consider*

$$\mathbf{x}A = (x, 1-x)\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = (x-(1-x), 1-x) = (2x-1, 1-x)$$

This shows that when the row player uses mixed strategy $\mathbf{x} = (x, 1-x)$, *then his payoff is* $2x-1$ *if the column player uses his 1st strategy scissors(S) and is* $1-x$ *if the column player uses his 2nd strategy rock(R). Now we solve the equation* $2x-1 = 1-x$ *and get* $x = \frac{2}{3}$. *One may see that if* $0 \le x < \frac{2}{3}$, *then* $2x-1 < x-1$ *and a rational column player would use his 1st strategy scissors(S) and the payoff to the row player would be* $2x-1 < \frac{1}{3}$. *On the other hand, if* $\frac{2}{3} < x \le 1$, *then* $2x-1 > 1-x$ *and a rational column player would use his 2nd strategy rock(R) and the payoff to the row player would be* $1-x < \frac{1}{3}$. *Now if* $x = \frac{2}{3}$, *that is if the row player uses the mixed strategy* $(\frac{2}{3}, \frac{1}{3})$, *then he may guarantee that his payoff is* $1-x = 2x-1 = \frac{1}{3}$ *no matter how the column player plays. This is the largest payoff he may guarantee and therefore the mixed strategy* $\mathbf{p} = (\frac{2}{3}, \frac{1}{3})$ *is the maximin strategy for the row player. Similarly, suppose the column player uses mixed strategy* $\mathbf{y} = (y, 1-y)$. *Consider*

$$A\mathbf{y}^T = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} y \\ 1-y \end{pmatrix} = \begin{pmatrix} y \\ -y+(1-y) \end{pmatrix}\begin{pmatrix} y \\ 1-2y \end{pmatrix}$$

If $0 \le y < \frac{1}{3}$, *then* $y < 1-2y$ *and a rational row player would use his 2nd strategy paper(P) and his payoff would be* $1-2y > \frac{1}{3}$. *If* $\frac{1}{3} < y \le 1$, *then* $y > 1-2y$ *and a rational row player would use his 1st strategy rock(R) and his payoff would be* $y > \frac{1}{3}$. *If* $y = \frac{1}{3}$, *then the payoff to the row player is always* $\frac{1}{3}$ *no matter how he plays. Therefore* $\mathbf{q} = (\frac{1}{3}, \frac{2}{3})$ *is the minimax strategy for the column player and he may guarantee that the payoff to the row player is* $\frac{1}{3}$ *no matter how the row player plays. Moreover the value of the game is* $\nu = \frac{1}{3}$. *We summarize the above discussion in the following statements.*

1. *The row player may use his maximin strategy* $\mathbf{p} = (\frac{2}{3}, \frac{1}{3})$ *to guarantee that his payoff is* $\nu = \frac{1}{3}$ *no matter how the column player plays.*

2. *The column player may use his minimax strategy* $\mathbf{q} = (\frac{1}{3}, \frac{2}{3})$ *to guarantee that the payoff to the row player is* $\nu = \frac{1}{3}$ *no matter how the row player plays.* □

Now we give the complete solutions to 2×2 games.

Theorem 1.2.2. *Let*

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

be a 2×2 game matrix. Suppose A has no saddle point. Then

1. *The value of the game is*
$$\nu = \frac{ad - bc}{a - b - c + d}$$
2. *The maximin strategy for the row player is*
$$\mathbf{p} = \left(\frac{d - c}{a - b - c + d}, \frac{a - b}{a - b - c + d} \right)$$
3. *The minimax strategy for the column player is*
$$\mathbf{q} = \left(\frac{d - b}{a - b - c + d}, \frac{a - c}{a - b - c + d} \right)$$

Proof. Suppose the row player uses mixed strategy $\mathbf{x} = (x, 1 - x)$. Consider

$$\mathbf{x}A = (x, 1-x) \begin{pmatrix} a & b \\ c & d \end{pmatrix} = (ax+c(1-x), bx+d(1-x)) = ((a-c)x+c, (b-d)x+d)$$

Now the payoff to the row player that he can guarantee is

$$\min\{(a - c)x + c, (b - d)x + d\}$$

Since A has no saddle point, we have $a - c$ and $b - d$ are of different sign and the maximum of the above minimum is obtained when

$$\begin{aligned} (a - c)x + c &= (b - d)x + d \\ \Rightarrow \qquad x &= \frac{d - c}{a - b - c + d} \end{aligned}$$

Note that x and $1 - x = \frac{a-b}{a-b-c+d}$ must be of the same sign because A has no saddle point. We must have $0 < x < 1$ and we conclude that the maximin strategy for the row player is

$$\mathbf{p} = \left(\frac{d - c}{a - b - c + d}, \frac{a - b}{a - b - c + d} \right)$$

Similarly suppose the column player uses mixed strategy $\mathbf{y} = (y, 1-y)$. Consider

$$A\mathbf{y}^T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} y \\ 1-y \end{pmatrix} = \begin{pmatrix} aq + b(1-y) \\ cq + d(1-y) \end{pmatrix} = \begin{pmatrix} (a-b)y + b \\ (c-d)y + d \end{pmatrix}$$

The column player may guarantee that the payoff to the row player is at most

$$\max\{(a-b)y + b, (c-d)y + d\}$$

The above quantity attains it minimum when

$$\begin{array}{rrcl} & (a-b)y + b & = & (c-d)y + d \\ \Rightarrow & y & = & \dfrac{d-b}{a-b-c+d} \end{array}$$

and the minimax strategy for the column player is

$$\mathbf{q} = \left(\frac{d-b}{a-b-c+d}, \frac{a-c}{a-b-c+d} \right)$$

By calculating

$$\mathbf{p}A = \left(\frac{ad-bc}{a-b-c+d}, \frac{ad-bc}{a-b-c+d} \right) \text{ and } A\mathbf{q}^T = \begin{pmatrix} \frac{ad-bc}{a-b-c+d} \\ \frac{ad-bc}{a-b-c+d} \end{pmatrix}$$

we see that the maximum payoff that the row player may guarantee to himself and the minimum payoff to the row player that the column player may guarantee are both $\frac{ad-bc}{a-b-c+d}$. In fact the minimax theorem (Theorem 1.1.10) says that these two values must be equal. We conclude that the value of A is $\nu = \frac{ad-bc}{a-b-c+d}$. □

Note that the above formulas work only when A has no saddle point. If A has a saddle point, the vectors $\mathbf{p}$ and ıı obtained using the formulas may not be probability vectors.

Example 1.2.3. *Consider the modified rock-paper-scissors game (Example 1.2.1) with game matrix*

$$A = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

The game matrix has no saddle point. By Theorem 1.2.2, the value of the game is

$$\nu = \frac{ad - bc}{a - b - c + d} = \frac{1 \times 1 - 0 \times (-1)}{1 - 0 - (-1) + 1} = \frac{1}{3}$$

the maximin strategy for the row player is

$$\begin{aligned} \mathbf{p} &= \left(\frac{d - c}{a - b - c + d}, \frac{a - b}{a - b - c + d} \right) \\ &= \left(\frac{1 - (-1)}{1 - 0 - (-1) + 1}, \frac{1 - 0}{1 - 0 - (-1) + 1} \right) \\ &= \left(\frac{2}{3}, \frac{1}{3} \right) \end{aligned}$$

and the minimax strategy for the column player is

$$\begin{aligned} \mathbf{q} &= \left(\frac{d - b}{a - b - c + d}, \frac{a - c}{a - b - c + d} \right) \\ &= \left(\frac{1 - 0}{1 - 0 - (-1) + 1}, \frac{1 - (-1)}{1 - 0 - (-1) + 1} \right) \\ &= \left(\frac{1}{3}, \frac{2}{3} \right) \end{aligned}$$

□

Example 1.2.4. *In a game, each of the two players Andy and Bobby calls out a number simultaneously. Andy may call out either* 1 *or* -2 *while Bobby may call out either* 1 *or* -3*. Then Bobby pays* p *dollars to Andy where* p *is the product of the two numbers (Andy pays Bobby* $-p$ *dollars when* p *is negative). The game matrix of the game is*

$$A = \begin{pmatrix} 1 & -3 \\ -2 & 6 \end{pmatrix}$$

The value of the game is

$$\nu = \frac{1 \times 6 - (-2) \times (-3)}{1 - (-3) - (-2) + 6} = 0$$

the maximin strategy for Andy is

$$\mathbf{p} = \left(\frac{6 - (-2)}{1 - (-3) - (-2) + 6}, \frac{1 - (-3)}{1 - (-3) - (-2) + 6} \right) = \left(\frac{2}{3}, \frac{1}{3} \right)$$

and the minimax strategy for Bobby is

$$\mathbf{q} = \left(\frac{6-(-3)}{1-(-3)-(-2)+6}, \frac{1-(-2)}{1-(-3)-(-2)+6} \right) = \left(\frac{3}{4}, \frac{1}{4} \right)$$

□

We say that a two-person zero sum game is **fair** if its value is zero. The game in Example 1.2.4 is a fair game.

1.3 Games reducible to 2×2 games

To solve an $m \times n$ game matrix for $m, n > 2$ without saddle point, we may first remove the dominated rows or columns. A row dominates another if all its entries are larger than or equal to the corresponding entries of the other. Similarly, a column dominates another if all its entries are smaller than or equal to the corresponding entries of the other.

Definition 1.3.1. *Let $A = [a_{ij}]$ be an $m \times n$ game matrix.*

1. *We say that the k-th row is dominated by the r-th row if $a_{kj} \leq a_{rj}$ for any $j = 1, 2, \cdots, n$.*
2. *We say that the l-th column is dominated the s-th column if $a_{il} \geq a_{is}$ for any $i = 1, 2, \cdots, m$.*

We say that a row (column) is a **dominated row (column)** *if it is dominated by another row (column).*

If the k-th row of A is dominated by the r-th row, then for the row player, playing the r-th strategy is at least as good as playing the k-th strategy. Thus the k-th row can be ignored in finding the maximin strategy for the row player. Similarly the column player may ignore a dominated column when finding his minimax strategy.

Theorem 1.3.2. *Let A be an $m \times n$ game matrix. Suppose A has a dominated row or dominated column. Let A' be the matrix obtained by deleting a dominated row or dominated column from A. Then*

1. *The value of A' is equal to the value of A.*

2. *The players of A have maximin/minimax strategies which never use dominated row/column.*

More precisely, if the k-th row is a dominated row of A, A' is the $(m-1)\times n$ matrix obtained by deleting the k-th row from A, and $\mathbf{p}' = (p_1, \cdots, p_{k-1}, p_{k+1}, \cdots, p_m) \in \mathcal{P}^{m-1}$ is a maximin strategy for the row player of A', then $\mathbf{p} = (p_1, \cdots, p_{k-1}, 0, p_{k+1}, \cdots, p_m) \in \mathcal{P}^m$ is a maximin strategy for the row player of A. Similarly, if the l-th column is a dominated row of A, A' is the $m \times (n-1)$ matrix obtained by deleting the l-th column from A, and $\mathbf{q}' = (q_1, \cdots, q_{l-1}, q_{l+1} \cdots, q_n) \in \mathcal{P}^{n-1}$ is a minimax strategy of A', then $\mathbf{q} = (q_1, \cdots, q_{l-1}, 0, q_{l+1}, \cdots, q_n) \in \mathcal{P}^n$ is a minimax strategy of A.

Proof. Suppose the k-th row of A is dominated by the r-th row and A' is obtained by deleting the k-th row from A. Let ν' be the value of A' and $\mathbf{q} \in \mathcal{P}^n$ be a minimax strategy of A'. For any mixed strategy $\mathbf{x} = (x_1, \cdots, x_m) \in \mathcal{P}^m$, define $\mathbf{x}' = (x'_1, \cdots, x'_{k-1}, x'_{k+1}, \cdots, x'_m) \in \mathcal{P}^{m-1}$ by

$$x'_i = \begin{cases} x_i & \text{if } i \neq r \\ x_k + x_r & \text{if } i = r \end{cases}$$

and we have

$$\mathbf{x}A\mathbf{q}^T \leq \mathbf{x}'A'\mathbf{q}^T \leq \nu'$$

Here the first inequality holds because the k-th is dominated by the r-th row and the second inequality holds because $\mathbf{q}$ is a minimax strategy of A'. Thus the value of A is less than or equal to ν'. On the other hand, let $\mathbf{p}' = (p_1, \cdots, p_{k-1}, p_{k+1}, \cdots, p_m) \in \mathcal{P}^{m-1}$ be a maximin strategy of A' and let $\mathbf{p} = (p_1, \cdots, p_{k-1}, 0, p_{k+1}, \cdots, p_m) \in \mathcal{P}^m$. Then we have

$$\mathbf{p}A\mathbf{y}^T = \mathbf{p}'A'\mathbf{y}^T \geq \nu'$$

for any $\mathbf{y} \in \mathcal{P}^n$. It follows that the value of A is ν' and $\mathbf{p}$ is a maximin strategy of A. The proof of the second statement is similar. □

The removal of dominated rows or columns does not change the value of a game. The above theorem only says that there is at least one optimal strategy with zero probability at the dominated rows and columns. There may be other optimal strategies which have positive probability at the dominated rows or columns. However any optimal strategy must have zero probability at strictly dominated rows and columns. Here a row is strictly dominated by another row if all its entries are strictly smaller than the corresponding entries of the other. Similarly a column is strictly dominated by another column if all its entries are strictly larger than the corresponding entries of the other.

Example 1.3.3. *To solve the game matrix*

$$A = \begin{pmatrix} 3 & -1 & 4 \\ 2 & -3 & 1 \\ -2 & 4 & 0 \end{pmatrix}$$

we may delete the second row since it is dominated by the first row and get the reduced matrix

$$A' = \begin{pmatrix} 3 & -1 & 4 \\ -2 & 4 & 0 \end{pmatrix}$$

Then we may delete the third column since is dominated by the first column. Hence the matrix A is reduced to the 2×2 matrix

$$A'' = \begin{pmatrix} 3 & -1 \\ -2 & 4 \end{pmatrix}$$

The value of this 2×2 matrix is 0.7. The maximin and minimax strategies are $(0.6, 0.4)$ and $(0.5, 0.5)$ respectively. Therefore the value of A is 0.7, a maximin strategy for the row player is $(0.6, 0, 0.4)$ and a minimax strategy for the column player is $(0.5, 0.5, 0)$. Note that we need to insert the zeros to the dominated rows and columns when writing down the maximin and minimax strategies for the players. □

1.4 $2 \times n$ and $m \times 2$ games

Let

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \end{pmatrix}$$

be a $2 \times n$ matrix. We are going to explain how to solve the game with game matrix A if there is no dominated row or column. Suppose the row player uses strategy $\mathbf{x} = (x, 1-x)$ for $0 \leq x \leq 1$. The payoff to the row player is given by

$$\begin{aligned} \mathbf{x}A &= (x, 1-x) \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \end{pmatrix} \\ &= (a_{11}x + a_{21}(1-x), a_{12}x + a_{22}(1-x), \cdots, a_{1n}x + a_{2n}(1-x)) \end{aligned}$$

Now we need to find the value of x so that the minimum

$$\min_{1\le j\le n}\{a_{1j}x + a_{2j}(1-x)\}$$

of the coordinates of $\mathbf{x}A$ attains its maximum. We may use graphical method to achieve this goal.

Step 1.
For each $1 \le j \le n$, draw the graph of

$$v = a_{1j}x + a_{2j}(1-x), \text{ for } 0 \le x \le 1$$

The graph shows the payoff to the row player if the column player uses the j-th strategy.

Step 2.
Draw the graph of

$$v = \min_{1\le j\le n}\{a_{1j}x + a_{2j}(1-x)\}$$

This is called the **lower envelope** of the graph.

Step 3.
Suppose (p, ν) is a maximum point of the lower envelope. Then ν is the value of the game and $\mathbf{p} = (p, 1-p)$ is a maximin strategy for the row player.

Step 4.
The solutions for $\mathbf{y} \in \mathcal{P}^n$ to the equation

$$A\mathbf{y}^T = \nu\mathbf{1}^T$$

where $\mathbf{1} = (1, 1)$, give the minimax strategy for the column player.

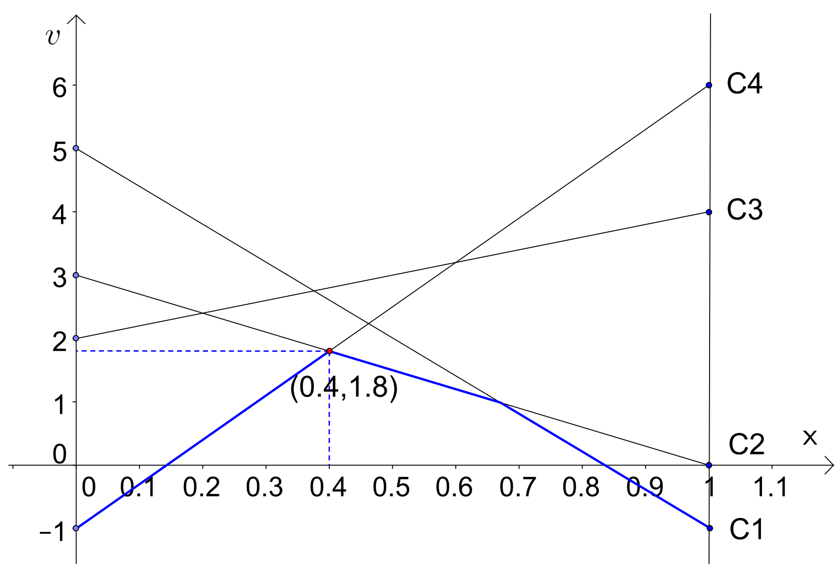
Example 1.4.1. *Solve the* 2×4 *game matrix*

$$A = \begin{pmatrix} -1 & 0 & 4 & 6 \\ 5 & 3 & 2 & -1 \end{pmatrix}$$

Solution.

Step 1. Draw the graph of

$$\begin{cases} C1: & v = -x + 5(1-x) \\ C2: & v = 3(1-x) \\ C3: & v = 4x + 2(1-x) \\ C4: & v = 6x - (1-x) \end{cases}$$



Step 2. Draw the lower envelope (blue polygonal curve).

Step 3. The maximum point of the lower envelope is the intersection point of $C2$ and $C4$. By solving

$$\begin{cases} C2: & v = 3(1-x) \\ C4: & v = 6x - (1-x) \end{cases}$$

we obtain the maximum point $(p, \nu) = (0.4, 1.8)$ of the lower envelope.

Step 4. Find the minimax strategies for the column player by solving

$$\begin{pmatrix} 0 & 6 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} y_2 \\ y_4 \end{pmatrix} = \begin{pmatrix} 1.8 \\ 1.8 \end{pmatrix}$$

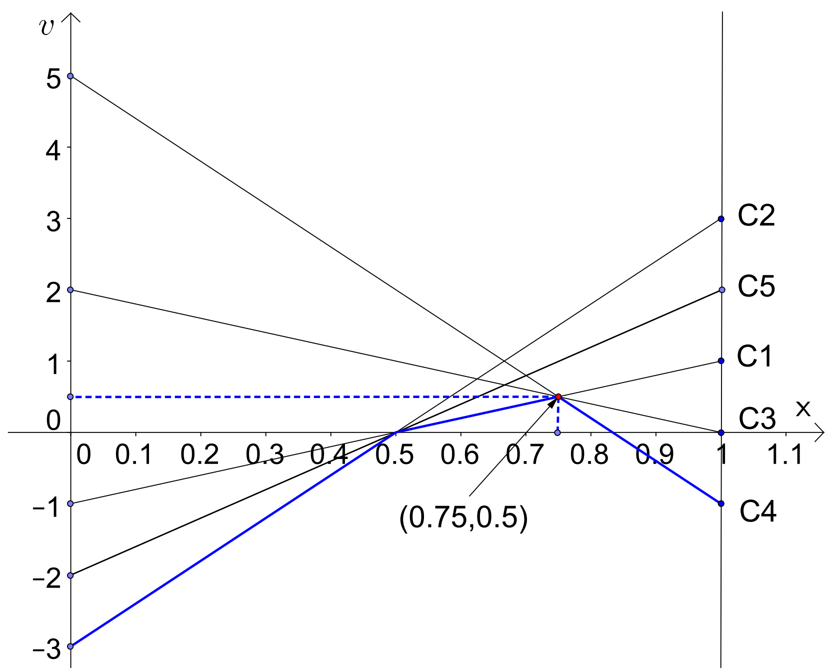
and get $y_2 = 0.7$ and $y_4 = 0.3$.

Therefore the value of the game is $\nu = 1.8$. The maximin strategy for the row player is $\mathbf{p} = (0.4, 0.6)$ and the minimax strategy for the column player is $\mathbf{q} = (0, 0.7, 0, 0.3)$. $\square$

Example 1.4.2. *Solve the* 2×5 *game matrix*

$$A = \begin{pmatrix} 1 & 3 & 0 & -1 & 2 \\ -1 & -3 & 2 & 5 & -2 \end{pmatrix}$$

Solution. The lower envelope is shown in the following figure.



By solving

$$\begin{cases} C1: & v = x - (1-x) \\ C3: & v = 2(1-x) \\ C4: & v = -x + 5(1-x) \end{cases}$$

we see that the maximum point of the lower envelope is $(p, \nu) = (0.75, 0.5)$. Thus the maximin strategy for the row player is $(0.75, 0.25)$ and the value of the game is $\nu = 0.5$. To find minimax strategies for the column player, we solve

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & 2 & 5 \end{pmatrix} \begin{pmatrix} y_1 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0.5 \\ 0.5 \end{pmatrix}$$

Note that we have added the equation $y_1+y_3+y_4 = 1$ to exclude the solutions which are not probability vectors. (Explain why we didn't do it in Example 1.4.1.) Using row operation, we obtain the row echelon form

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 0 & -1 & 0.5 \\ -1 & 2 & 5 & 0.5 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0.5 \\ 0 & 1 & 2 & 0.5 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

The non-negative solution to the system of equations is

$$(y_1, y_3, y_4) = (0.5 + t, 0.5 - 2t, t) \text{ for } 0 \leq t \leq 0.25$$

Therefore the column player has minimax strategies

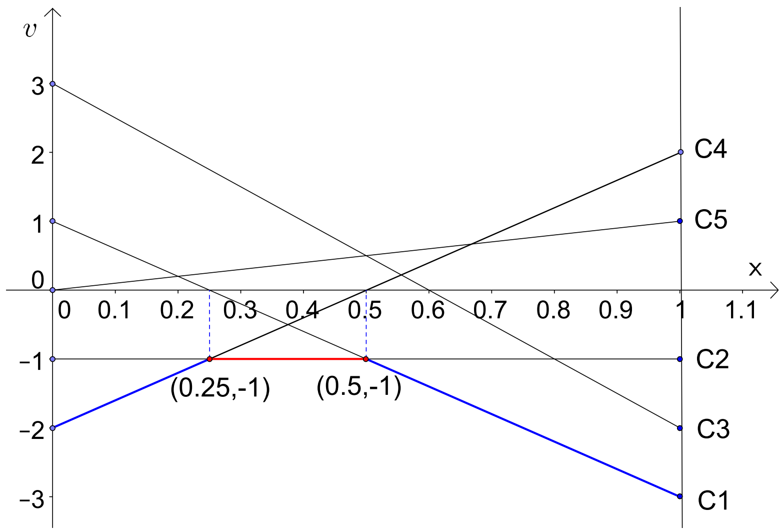
$$\mathbf{q} = (0.5 + t, 0, 0.5 - 2t, t, 0) \text{ for } 0 \leq t \leq 0.25$$

In particular, $(0.5, 0, 0.5, 0, 0)$ and $(0.75, 0, 0, 0.25, 0)$ are minimax strategies for the column player. □

Example 1.4.3. *Solve the* 2×5 *game matrix*

$$A = \begin{pmatrix} -3 & -1 & -2 & 2 & 1 \\ 1 & -1 & 3 & -2 & 0 \end{pmatrix}$$

Solution. The lower envelope is shown in the following figure.



The maximum points of the lower envelope are points lying on the line segment joining $(0.25, -1)$ and $(0.5, -1)$. Thus the value of the game is $\nu = -1$. The maximin strategies for the row player are

$$\mathbf{p} = (p, 1-p) \text{ for } 0.25 \leq p \leq 0.5$$

and the minimax strategy for the column player is

$$\mathbf{q} = (0, 1, 0, 0, 0)$$

□

Next we consider $m \times 2$ games. There are two methods to solve such games.

Method 1.
Let $\mathbf{y} = (y, 1-y)$, $0 \leq y \leq 1$, be the strategy for the column player. Draw the upper envelope

$$v = \max_{1 \leq i \leq m} \{a_{i1}y + a_{i2}(1-y)\}$$

Suppose the minimum point of the upper envelope is (q, ν). Then the value of the game is ν and the minimax strategy for the column player is $\mathbf{q} = (q, 1-q)$. Moreover the maximum strategies for the row player are the solutions for $\mathbf{x} \in \mathcal{P}^m$ to the equation

$$\mathbf{x}A = \nu\mathbf{1} = (\nu, \nu)$$

Method 2.
Solve the game with $2 \times m$ game matrix $-A^T$. Then

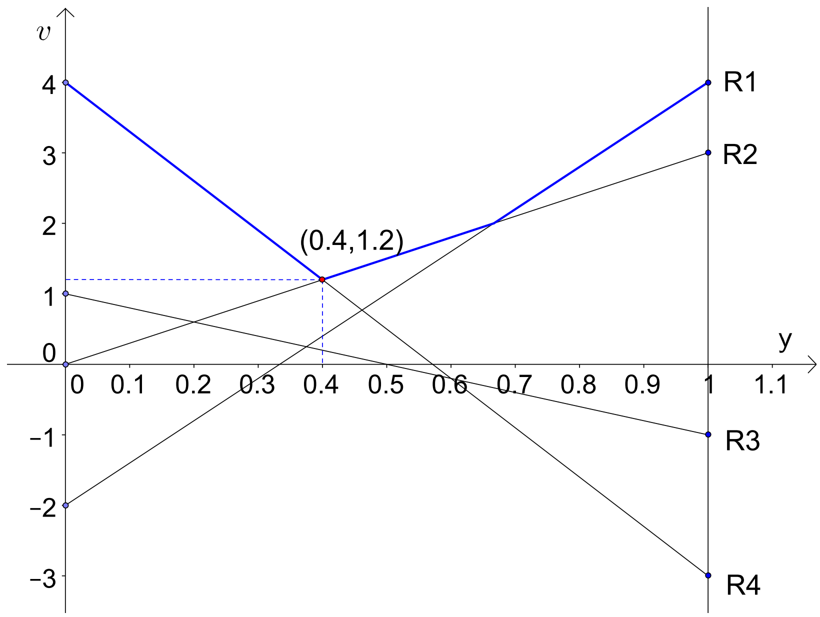
value of $A = -$ value of $-A^T$
maximin strategy of $A =$ minimax strategy of $-A^T$
minimax strategy of $A =$ maximin strategy of $-A^T$

Example 1.4.4. *Solve the* 4×2 *game matrix*

$$A = \begin{pmatrix} 4 & -2 \\ 3 & 0 \\ -1 & 1 \\ -3 & 4 \end{pmatrix}$$

Solution.

Method 1.
Let $\mathbf{y} = (y, 1-y)$, $0 \leq y \leq 1$, be the strategy of the column player. The upper envelope is



Solving

$$\begin{cases} R2: & v = 3(1-y) \\ R4: & v = -3y + 4(1-y) \end{cases}$$

the minimum point of the upper envelope is $(q, \nu) = (0.4, 1.2)$. Now the row player would only use the 2nd and 4th strategy and we solve

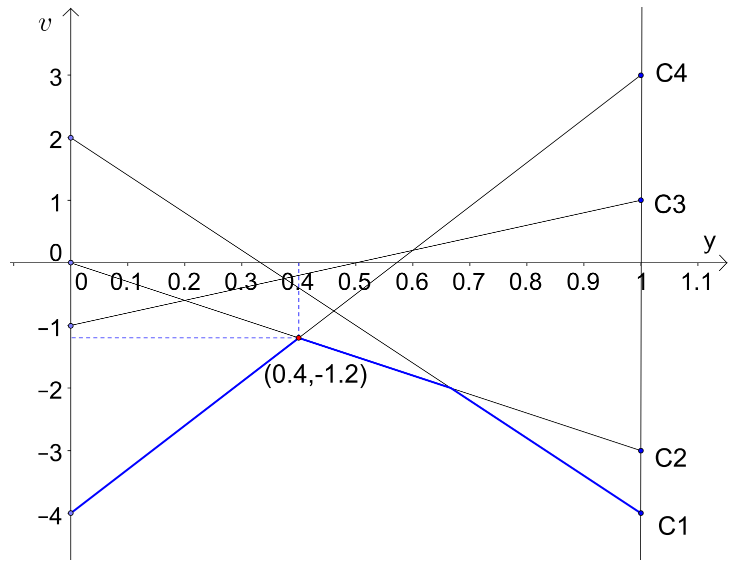
$$(x_2, x_4) \begin{pmatrix} 3 & 0 \\ -3 & 4 \end{pmatrix} = (1.2, 1.2)$$

which gives $(x_2, x_4) = (0, 7, 0.3)$. Therefore the value of the game is $\nu = 1.2$, the maximin strategy for the row player is $\mathbf{p} = (0, 0.7, 0, 0.3)$ and the minimax strategy for the column player is $\mathbf{q} = (0.4, 0.6)$.

Method 2.
Consider

$$-A^T = \begin{pmatrix} -4 & -3 & 1 & 3 \\ 2 & 0 & -1 & -4 \end{pmatrix}$$

Draw the lower envelope



We see that the value of $-A^T$ is -1.2 and the maximin strategy of $-A^T$ is $(0.4, 0.6)$. Solving

$$\begin{pmatrix} -3 & 3 \\ 0 & -4 \end{pmatrix} \begin{pmatrix} x_2 \\ x_4 \end{pmatrix} = \begin{pmatrix} -1.2 \\ -1.2 \end{pmatrix}$$

We get $x_2 = 0.7$ and $x_4 = 0.3$. Thus the minimax strategy of $-A^T$ is $(0, 0.7, 0, 0.3)$. Therefore

$$\begin{aligned}
&\text{value of } A = -\text{ value of } -A^T = 1.2 \\
&\text{maximin strategy of } A = \text{minimax strategy of } -A^T = (0, 0.7, 0, 0.3) \\
&\text{minimax strategy of } A = \text{maximin strategy of } -A^T = (0.4, 0.6)
\end{aligned}$$

□

Theorem 1.4.5 (Principle of indifference)**.** *Let A be an $m \times n$ game matrix. Suppose ν is the value of A, $\mathbf{p} = (p_1, \cdots, p_m)$ be a maximin strategy for the row player and $\mathbf{q} = (q_1, \cdots, q_n)$ be a minimax strategy for the column player. For any $k = 1, 2, \cdots, m$, if $p_k > 0$, then $\sum_{j=1}^{n} a_{kj} q_j = \nu$. In particular, when the column player uses his minimax strategy $\mathbf{q}$, then the payoff to the row* *player are indifferent among all his k-th strategies with $p_k > 0$. Similarly, for any $l = 1, 2, \cdots, n$, if $q_l > 0$, then $\sum_{i=1}^{m} a_{il} p_i = \nu$. In particular, when the row player uses his maximin strategy* $\mathbf{p}$*, then the payoff to the row player are indifferent among all the l-th strategies of the column player with $q_l > 0$.*

Proof. For each $k = 1, 2, \cdots, m$, we have

$$\sum_{j=1}^{n} a_{kj} q_j \leq \nu$$

since $\mathbf{q}$ is a minimax strategy for the column player. On the other hand,

$$\nu = \mathbf{p} A \mathbf{q}^T = \sum_{k=1}^{m} p_k \left(\sum_{j=1}^{n} a_{kj} q_j \right) \leq \sum_{k=1}^{m} p_k \nu = \nu$$

Thus we have

$$p_k \sum_{j=1}^{n} a_{kj} q_j = p_k \nu$$

for any $k = 1, 2, \cdots, m$. Therefore

$$\sum_{j=1}^{n} a_{kj} q_j = \nu$$

whenever $p_k > 0$. The proof of the second statement is similar. $\square$

Exercise 1

1. Find the values of the following game matrices by finding their saddle points

   (a) $\begin{pmatrix} 5 & 1 & -2 & 6 \\ -1 & 0 & 1 & -2 \\ 3 & 2 & 5 & 4 \end{pmatrix}$ (b) $\begin{pmatrix} -4 & 5 & -3 & -3 \\ 0 & 1 & 3 & -1 \\ -3 & -1 & 2 & -5 \\ 2 & -4 & 0 & -2 \end{pmatrix}$

2. Solve the following game matrix, that is, find the value of the game, a maximin strategy for the row player and a minimax strategy for the column.

(a) $\begin{pmatrix} 1 & 7 \\ 2 & -2 \end{pmatrix}$

(b) $\begin{pmatrix} 3 & -1 \\ -2 & 4 \end{pmatrix}$

(c) $\begin{pmatrix} 3 & 2 & 4 & 0 \\ -2 & 1 & -4 & 5 \end{pmatrix}$

(d) $\begin{pmatrix} 1 & 0 & 4 & 2 \\ 0 & 2 & -3 & -2 \end{pmatrix}$

(e) $\begin{pmatrix} 5 & -3 \\ -3 & 5 \\ 2 & -1 \\ 4 & 0 \end{pmatrix}$

(f) $\begin{pmatrix} 5 & -2 & 4 \\ 3 & -3 & 1 \\ 0 & 3 & 2 \end{pmatrix}$

(g) $\begin{pmatrix} 5 & 1 & -2 & 6 \\ -1 & 0 & 1 & -2 \\ 3 & 2 & 5 & 4 \end{pmatrix}$

3. Raymond holds a black 2 and a red 9. Calvin holds a red 3 and a black 8. Each of them chooses one of the cards from his hand and then two players show the chosen cards simultaneously. If the chosen cards are of the same colour, Raymond wins and Calvin wins if the cards are of different colours. The loser pays the winner an amount equal to the number on the winner's card. Write down the game matrix, find the value of the game and the optimal strategies of the players.

4. Alex and Becky point fingers to each other, with either one finger or two fingers. If they match with one finger, Becky pays Alex 3 dollars. If they match with two fingers, Becky pays Alex 11 dollars. If they don't match, Alex pays Becky 1 dollar.

   (a) Find the optimal strategies for Alex and Becky.

   (b) Suppose Alex pays Becky k dollars as a compensation before the game. Find the value of k to make the game fair.

5. Player I and II choose integers i and j respectively where $1 \leq i, j \leq 7$. Player II pays Player I one dollar if $|i - j| = 1$. Otherwise there is no payoff. Write down the game matrix of the game, find the value of the game and the optimal strategies for the players.

6. Use the principle of indifference to solve the game with game matrix

$$\begin{pmatrix} 1 & -2 & 3 & -4 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

7. In the Mendelsohn game, two players choose an integer from 1 to 5 simultaneously. If the numbers are equal there is no payoff. The player that chooses a number one larger than that chosen by his opponent wins 1 dollar from its opponent. The player that chooses a number two or more larger than his opponent loses 2 dollars to its opponent. Find the game matrix and solve the game.

8. Aaron puts a chip in either his left hand or right hand. Ben guesses where the chip is. If Ben guesses the left hand, he receives \$2 from Aaron if he is correct and pays \$4 to Aaron if he is wrong. If Ben guesses the right hand, he receives \$1 from Aaron if he is correct and pays \$3 to Aaron if he is wrong.

   (a) Write down the payoff matrix of Aaron. (Use order of strategies: Left, Right.)

   (b) Find the maximin strategy for Aaron, the minimax strategy for Ben and the value of the game.

9. Let
$$A = \begin{pmatrix} -3 & 1 \\ c & -2 \end{pmatrix}$$
   where c is a real number.

   (a) Find the range of values of c such that A has a saddle point.

   (b) Suppose the zero sum game with game matrix A is a fair game.

      (i) Find the value of c.

      (ii) Find the maximin strategy for the row player and the minimax strategy for the column player.

10. Prove that if A is a skewed symmetric matrix, that is, $A^T = -A$, then the value of A is zero.

11. Let $\mathbf{1} = (1, 1, \cdots, 1)$. Prove the following statements.

   (a) If A is a symmetric matrix, that is $A^T = A$, and there exists probability vector $\mathbf{y} \in \mathcal{P}^n$ such that $A\mathbf{y}^T = v\mathbf{1}^T$ where $v \in \mathbb{R}$ is a real number, then v is the value of A.

   (b) There exists a square matrix A, a probability vector $\mathbf{y}$ and a real number v such that $A\mathbf{y}^T = v\mathbf{1}^T$ but v is not the value of A.

12. Let n be a positive integer and

$$D = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}$$

be an $n \times n$ diagonal matrix where $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$.

(a) Suppose $\lambda_1 \leq 0$ and $\lambda_n > 0$. Find the value of the zero sum game with game matrix D.

(b) Suppose $\lambda_1 > 0$. Solve the zero sum game with game matrix D.

13. Let

$$A = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix}.$$

(a) Find a vector $\mathbf{x} = (1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5$ and a real number a such that

$$A\mathbf{x}^T = (0, 0, 0, 0, a)^T$$

(b) Find a vector $\mathbf{y} = (1, y_2, y_3, y_4, y_5) \in \mathbb{R}^5$ and a real number b such that

$$A\mathbf{y}^T = (1, 1, 1, 1, b)^T$$

(c) Find the maximin strategy, the minimax strategy and the value of A. (Hint: Find real numbers $\alpha, \beta \in \mathbb{R}$ such that $\mathbf{q} = \alpha\mathbf{x} + \beta\mathbf{y}$ satisfies $A\mathbf{q}^T = v\mathbf{1}^T$ for some $v \in \mathbb{R}$.)

14. For positive integer k, define

$$A_k = \begin{pmatrix} 4k-3 & -(4k-2) \\ -(4k-1) & 4k \end{pmatrix}.$$

(a) Solve A_k, that is, find the maximin strategy, minimax strategy and value of A_k in terms of k.

(b) Let $r_1, r_2, \cdots, r_n > 0$ be positive real numbers. Using the principle of indifference, or otherwise, find, in terms of $r_1, r_2, \cdots, r_n$, the value of

$$D = \begin{pmatrix} \frac{1}{r_1} & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{r_2} & 0 & \cdots & 0 \\ 0 & 0 & \frac{1}{r_3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{r_n} \end{pmatrix}.$$

(c) Find, with proof, the value of the matrix

$$A = \begin{pmatrix} A_1 & 0 & 0 & \cdots & 0 \\ 0 & A_2 & 0 & \cdots & 0 \\ 0 & 0 & A_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & A_{25} \end{pmatrix}.$$

2 Linear programming and maximin theorem

2.1 Linear programming

In this chapter we study two-person zero sum game with $m \times n$ game matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

Suppose the row player uses strategy $\mathbf{x} = (x_1, \cdots, x_m) \in \mathcal{P}^m$. Then the column player would use his j-th strategy such that

$$a_{1j}p_1 + a_{2j}p_2 + \cdots + a_{mj}p_m$$

is minimum among $j = 1, 2, \cdots, n$. Thus the payoff to the row player that he can guarantee is

$$\min_{j=1,2,\cdots,n} \{a_{1j}x_1 + a_{2j}x_2 + \cdots + a_{mj}x_m\}$$

Hence if the above expression attains its maximum at $\mathbf{x} = \mathbf{p} \in \mathcal{P}^m$, then $\mathbf{p}$ is a maximin strategy for the row player. Moreover, the value of the game is

$$\nu = \max_{\mathbf{x} \in \mathcal{P}^m} \min_{j=1,2,\cdots,n} \{a_{1j}x_1 + a_{2j}x_2 + \cdots + a_{mj}x_m\}$$

By introducing a new variable v, we can restate the **maximin problem**, that is finding a maximin strategy, as the following linear programming problem

$$\begin{array}{rl} \max & v \\ \text{subject to} & a_{11}p_1 + a_{21}p_2 + \cdots + a_{m1}p_m \geq v \\ & a_{12}p_1 + a_{22}p_2 + \cdots + a_{m2}p_m \geq v \\ & \vdots \\ & a_{1n}p_1 + a_{2n}p_2 + \cdots + a_{mn}p_m \geq v \\ & p_1 + p_2 + \cdots + p_m = 1 \\ & p_1, p_2, \cdots, p_m \geq 0 \end{array}$$

Similarly, to find a minimax strategy for the column player, we need to solve the following **minimax** problem

$$\begin{array}{rl} \min & v \\ \text{subject to} & a_{11}q_1 + a_{12}q_2 + \cdots + a_{1n}q_n \leq v \\ & a_{21}q_1 + a_{22}q_2 + \cdots + a_{2n}q_n \leq v \\ & \vdots \\ & a_{m1}q_1 + a_{m2}q_2 + \cdots + a_{mn}q_n \leq v \\ & q_1 + q_2 + \cdots + q_n = 1 \\ & q_1, q_2, \cdots, q_n \geq 0 \end{array}$$

To solve the maximin and minimax problems, first we transform them to a pair of primal and dual problems.

Definition 2.1.1 (Primal and dual problems)**.** *A linear programming problem in the following form is called a* **primal problem***.*

$$\begin{array}{rl} \max & f(y_1, \cdots, y_n) = \sum_{j=1}^{n} c_j y_j + d \\ \textit{subject to} & \sum_{j=1}^{n} a_{ij} y_j \leq b_i, \ i = 1, 2, \cdots, m \\ & y_1, y_2, \cdots, y_n \geq 0 \end{array}$$

The **dual problem** *associated to the above primal problem is*

$$\begin{array}{rl} \min & g(x_1, \cdots, x_m) = \sum_{i=1}^{m} b_i x_i + d \\ \textit{subject to} & \sum_{i=1}^{m} a_{ij} x_i \geq c_j, \ j = 1, 2, \cdots, n \\ & x_1, x_2, \cdots, x_m \geq 0 \end{array}$$

Here $x_1, \cdots, x_m, y_1, \cdots, y_n$ are variables, and a_{ij}, b_i, c_j, d, $i = 1, 2, \cdots, m$, $j = 1, 2, \cdots, n$, are constants. The linear functions f and g are called **objective functions***. The primal problem and the dual problem can be written in the following matrix forms*

Primal problem	max *subject to*	$f(\mathbf{y}) = \mathbf{c}\mathbf{y}^T + d$ $A\mathbf{y}^T \leq \mathbf{b}^T$ $\mathbf{y} \geq \mathbf{0}$
Dual problem	min *subject to*	$g(\mathbf{x}) = \mathbf{x}\mathbf{b}^T + d$ $\mathbf{x}A \geq \mathbf{c}$ $\mathbf{x} \geq \mathbf{0}$

Here $\mathbf{x} \in \mathbb{R}^m$, $\mathbf{y} \in \mathbb{R}^n$ *are variable vectors,* A *is an* $m \times n$ *constant matrix,* $\mathbf{b} \in \mathbb{R}^m$, $\mathbf{c} \in \mathbb{R}^n$ *are constant vectors and* $d \in \mathbb{R}$ *is a real constant. The inequality* $\mathbf{u} \leq \mathbf{v}$ *for vectors* $\mathbf{u}, \mathbf{v}$ *means each of the coordinates of* $\mathbf{v} - \mathbf{u}$ *is non-negative.*

For primal and dual problems, we always have the constraints $\mathbf{x}, \mathbf{y} \geq \mathbf{0}$. In other words, all variables are non-negative. From now on, we will not write down the constraints $\mathbf{x}, \mathbf{y} \geq \mathbf{0}$ for primal and dual problems and it is understood that all variables are non-negative.

Definition 2.1.2. *Suppose we have a pair of primal and dual problems.*

1. *We say that a vector* $\mathbf{x} \in \mathbb{R}^m$ *in the dual problem, (or* $\mathbf{y} \in \mathbb{R}^n$ *in the primal problem), is* **feasible** *if it satisfies the constraints of the problem. We say that the primal problem (or the dual problem) is feasible there exists a feasible vector for the problem.*

2. *We say that the primal problem, (or the dual problem), is* **bounded** *if the objective function is bounded above, (or below) on the set of feasible vectors.*

3. *We say that a feasible vector* $\mathbf{x} \in \mathbb{R}^m$ *in the dual problem, (or* $\mathbf{y} \in \mathbb{R}^n$ *in the primal problem), is* **optimal** *if the objective function* f *(or* g*) attains its maximum (or minimax) at* $\mathbf{x}$ *(or* $\mathbf{y}$*) on the set of feasible vectors.*

Theorem 2.1.3. *Suppose* $\mathbf{x}$ *and* $\mathbf{y}$ *are feasible vectors in the dual and primal problems respectively. Then*

$$f(\mathbf{y}) \leq g(\mathbf{x})$$

Proof. We have

$$\begin{aligned} f(\mathbf{y}) &= \mathbf{c}\mathbf{y}^T + d \\ &\leq \mathbf{x}A\mathbf{y}^T + d \quad \text{(since } \mathbf{x} \text{ is feasible and } \mathbf{y} \geq \mathbf{0}\text{)} \\ &\leq \mathbf{x}\mathbf{b}^T + d \quad \text{(since } \mathbf{y} \text{ is feasible and } \mathbf{x} \geq \mathbf{0}\text{)} \\ &= g(\mathbf{x}) \end{aligned}$$

□

The theorem above has a simple and important consequence that the primal problem is bounded if the dual problem associated with it has a feasible vector, and vice verse.

Theorem 2.1.4. *Suppose we have a pair of primal and dual problems.*

1. *If the primal problem is feasible, then the dual problem is bounded.*
2. *If the dual problem is feasible, then the primal problem is bounded.*
3. *If both problems are feasible, then both problems are solvable, that is, there exists optimal vectors* $\mathbf{p}$ *and* $\mathbf{q}$ *for the dual and primal problems respectively. Moreover we have* $f(\mathbf{p}) \leq g(\mathbf{q})$.

Proof. For the first statement, suppose the primal problem has a feasible vector $\mathbf{q}$. Then for any feasible vector $\mathbf{x}$ of the dual problem, we have $g(\mathbf{x}) \geq f(\mathbf{q})$ by Theorem 2.1.3. Hence the dual problem is bounded. The proof of the second statement is similar. For the third statement, suppose both problems are feasible. Then both problems are bounded by the first two statements. Observe that the set of feasible vectors is closed. It follows that the optimal values of the objective functions f and g are attainable. Therefore there exists optimal vectors $\mathbf{p}$ and $\mathbf{q}$ for the dual and primal problems respectively and $f(\mathbf{q}) \leq g(\mathbf{p})$ by Theorem 2.1.3. □

Furthermore we have the following important theorem in linear programming concerning the solutions to the primal and dual problems.

Theorem 2.1.5. *Suppose both the dual problem and the primal problem are feasible. Then there exist optimal vectors* $\mathbf{p}$ *and* $\mathbf{q}$ *for the dual and primal problem respectively, and we have*

$$f(\mathbf{q}) = g(\mathbf{p})$$

Proof. We have proved the solvability of the problems. The equality $f(\mathbf{q}) = g(\mathbf{p})$ can be proved using minimax theorem and we omit the proof here. □

2.2 Transforming maximin problem to dual problem

To find a maximin strategy for the row player of a two-person zero sum game, we have seen in the previous section that we need to solve the following maximin problem.

$$\begin{array}{rl} \max & v \\ \text{subject to} & a_{11}p_1 + a_{21}p_2 + \cdots + a_{m1}p_m \geq v \\ & a_{12}p_1 + a_{22}p_2 + \cdots + a_{m2}p_m \geq v \\ & \vdots \\ & a_{1n}p_1 + a_{2n}p_2 + \cdots + a_{mn}p_m \geq v \\ & p_1 + p_2 + \cdots + p_m = 1 \\ & p_1, p_2, \cdots, p_m \geq 0 \end{array}$$

which can be written into following matrix form

$$\begin{array}{rl} \max & v \\ \text{subject to} & \mathbf{p}A \geq v\mathbf{1} \\ & \mathbf{p}\mathbf{1}^T = 1 \\ & \mathbf{p} \geq \mathbf{0} \end{array}$$

where $\mathbf{1} = (1, \cdots, 1) \in \mathbb{R}^m$. We solve the above maximin problem in the following two steps.

1. Transform the maximin problem to a dual problem.
2. Use simplex method to solve the dual problem.

In this section, we are going to discuss how to transform a maximin problem to a dual problem. Note that the maximin problem is neither a primal nor dual problem because there is a constraint $p_1+p_2+\cdots+p_m = 1$ which is not allowed and we do not have the constraint $v \geq 0$. To transform the maximin problem into a dual problem, first we add a constant k to each entry of A so that the value of the game matrix is positive. Secondly, we let

$$x_i = \frac{p_i}{v}, \text{ for } i = 1, 2, \cdots, m$$

Then to maximize v is the same as minimizing

$$x_1 + x_2 + \cdots + x_m = \frac{p_1 + p_2 + \cdots + p_m}{v} = \frac{1}{v}$$

Moreover for each $j = 1, 2, \cdots, n$, the constraint

$$a_{1j}p_1 + a_{2j}p_2 + \cdots + a_{mj}p_m \geq v$$

is equivalent to

$$a_{1j}x_1 + a_{2j}x_2 + \cdots + a_{mj}x_m \geq 1$$

and the maximin problem would become a dual problem. We summarize the above procedures as follows.

1. First, add a constant k to each entry of A so that every entry of A is positive. (This is done to make sure that the value of the game matrix is positive.)

2. Let
$$x_i = \frac{p_i}{v}, \text{ for } i = 1, 2, \cdots, m$$

3. Write down the dual problem
$$\begin{array}{rl} \min & g(x_1, x_2, \cdots, x_m) = x_1 + x_2 + \cdots + x_m \\ \text{subject to} & a_{11}x_1 + a_{21}x_2 + \cdots + a_{m1}x_m \geq 1 \\ & a_{12}x_1 + a_{22}x_2 + \cdots + a_{m2}x_m \geq 1 \\ & \vdots \\ & a_{1n}x_1 + a_{2n}x_2 + \cdots + a_{mn}x_m \geq 1 \end{array}$$
(Note that we always have the constraints $x_1, x_2, \cdots, x_m \geq 0$) or in matrix form
$$\begin{array}{rl} \min & g(\mathbf{x}) = \mathbf{x}\mathbf{1}^T \\ \text{subject to} & \mathbf{x}A \geq \mathbf{1} \end{array}$$
where $\mathbf{1} = (1, 1 \cdots, 1) \in \mathbb{R}^m$.

4. Suppose $\mathbf{x} = (x_1, x_2, \cdots, x_m)$ is an optimal vector of the dual problem and
$$d = g(\mathbf{x}) = x_1 + x_2 + \cdots + x_m$$
is the minimum value. Then
$$\mathbf{p} = \frac{\mathbf{x}}{d} = \left(\frac{x_1}{d}, \frac{x_2}{d}, \cdots, \frac{x_m}{d}\right)$$
is a maximin strategy for the row player and the value of the game matrix A is
$$v = \frac{1}{d} - k$$

To find the minimax strategy for the column player, we need to solve the following minimax problem.

$$\begin{array}{rl} \min & v \\ \text{subject to} & A\mathbf{q}^T \leq v\mathbf{1}^T \\ & \mathbf{1}\mathbf{q}^T = 1 \\ & \mathbf{q} \geq \mathbf{0} \end{array}$$

where $\mathbf{1} = (1, \cdots, 1) \in \mathbb{R}^n$. If we assume that $v > 0$, the above optimization problem can be transformed to the following primal problem by taking $y_j = \frac{q_j}{v}$ for $j = 1, 2, \cdots, n$.

$$\begin{array}{rl} \max & f(\mathbf{y}) = \mathbf{1}\mathbf{y}^T \\ \text{subject to} & A\mathbf{y} \leq \mathbf{1}^T \end{array}$$

where $\mathbf{y} = (y_1, y_2, \cdots, y_n)$. (Note that we always have the constraint $\mathbf{y} \geq \mathbf{0}$ for primal problem.) Suppose $\mathbf{y}$ is an optimal vector for the above primal problem. Then $\mathbf{q} = \frac{\mathbf{y}}{d}$ is a minimax strategy for the column player.

2.3 Simplex method

We have seen that a pair of maximin and minimax problems can be transformed to a pair of dual and primal problems. In this section, we will show how to use simplex method to solve the dual and primal problems simultaneously. Recall that the primal and dual problems are optimization problems of the following forms. Please be reminded that we always have the constraints $\mathbf{x}, \mathbf{y} \geq \mathbf{0}$.

Primal problem	max subject to	$f(\mathbf{y}) = \mathbf{c}\mathbf{y}^T + d$ $A\mathbf{y}^T \leq \mathbf{b}^T$
Dual problem	min subject to	$g(\mathbf{x}) = \mathbf{x}\mathbf{b}^T + d$ $\mathbf{x}A \geq \mathbf{c}$

We describe the **simplex method** as follows.

Step 1. Introduce new variables $x_{m+1}, \cdots, x_{m+m}, y_{n+1}, \cdots, y_{n+m}$ which are called **slack variables** and set up the tableau

$$
\begin{array}{c|ccc|c|l}
 & y_1 & \cdots & y_n & -1 & \\
\hline
x_1 & a_{11} & \cdots & a_{1n} & b_1 & = -y_{n+1} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \quad\vdots \\
x_m & a_{m1} & \cdots & a_{mn} & b_m & = -y_{n+m} \\
\hline
-1 & c_1 & \cdots & c_n & -d & = f \\
 & \shortparallel & \cdots & \shortparallel & \shortparallel & \\
 & x_{m+1} & \cdots & x_{m+n} & g &
\end{array}
$$

Step 2.

(i) If $c_1, c_2, \cdots, c_n \leq 0$, then the solution to the problems are

Primal problem	maximum value of $f = d$ $y_1 = y_2 = \cdots = y_n = 0$, $y_{n+1} = b_1, y_{n+2} = b_2, \cdots, y_{n+m} = b_m$
Dual problem	minimum value of $g = d$ $x_1 = x_2 = \cdots = x_m = 0$, $x_{m+1} = -c_1, x_{m+2} = -c_2, \cdots, x_{m+n} = -c_m$

(ii) Otherwise go to step 3.

Step 3. Choose $l = 1, 2, \cdots, n$ such that $c_l > 0$.

(i) If $a_{il} \leq 0$ for all $i = 1, 2, \cdots, m$, then the problems are unbounded (because y_l can be arbitrarily large) and there is no solution.

(ii) Otherwise choose $k = 1, 2, \cdots, m$, such that

$$\frac{b_k}{a_{kl}} = \min_{a_{il} > 0} \left\{ \frac{b_i}{a_{il}} \right\}$$

Step 4. Pivot on the entry a_{kl} and swap the variables at the pivot row with the variables at the pivot column. The **pivoting operation** is performed as follows.

$$
\begin{array}{c|cc|l}
 & y_l & y_j & \\
\hline
x_k & a^* & b & = -y_{n+k} \\
x_i & c & d & = -y_{n+i} \\
\hline
 & \shortparallel & \shortparallel & \\
 & x_{m+l} & x_{m+j} &
\end{array}
\longrightarrow
\begin{array}{c|cc|l}
 & y_{n+k} & y_j & \\
\hline
x_{m+l} & \frac{1}{a} & \frac{b}{a} & = -y_l \\
x_i & -\frac{c}{a} & d - \frac{bc}{a} & = -y_{n+i} \\
\hline
 & \shortparallel & \shortparallel & \\
 & x_k & x_{m+j} &
\end{array}
$$

Step 5. Go to Step 2.

To understand how the simplex method works, we introduce basic forms of linear programming problem.

Definition 2.3.1 (Basic form). *A* **basic form** *of a pair of primal and dual problems is a problem of the form*

Primal basic form	max *subject to*	$f(\mathbf{y}) = \mathbf{c}\mathbf{y}^T + d$ $A\mathbf{y}^T - \mathbf{b}^T = -(y_{n+1}, \cdots, y_{n+m})^T$ $\mathbf{y} \geq \mathbf{0}$
Dual basic form	min *subject to*	$g(\mathbf{x}) = \mathbf{x}\mathbf{b}^T + d$ $\mathbf{x}A - \mathbf{c} = (x_{m+1}, \cdots, x_{m+n})$ $\mathbf{x} \geq \mathbf{0}$

where $\mathbf{x} = (x_1, \cdots, x_m) \in \mathbb{R}^m$ *and* $\mathbf{y} = (y_1, \cdots, y_n) \in \mathbb{R}^n$. *The pair of basic forms can be represented by the tableau*

$$
\begin{array}{c|ccc|c|l}
 & y_1 & \cdots & y_n & -1 & \\
\hline
x_1 & a_{11} & \cdots & a_{1n} & b_1 & = -y_{n+1} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \quad\vdots \\
x_m & a_{m1} & \cdots & a_{mn} & b_m & = -y_{n+m} \\
\hline
-1 & c_1 & \cdots & c_n & -d & = f \\
 & \shortparallel & \cdots & \shortparallel & \shortparallel & \\
 & x_{m+1} & \cdots & x_{m+n} & g &
\end{array}
$$

The variables at the rightmost column and at the bottom row are called **basic variables**. *The other variables at the leftmost columns and at the top row are called* **independent/non-basic variables**.

A pair of primal and dual problems may be expressed in basic form in many different ways. The pivot operation changes one basic form of the pair of primal and dual problems to another basic form of the same pair of problems, and swaps one basic variable with one independent variable.

Theorem 2.3.2. *The basic forms before and after a pivot operation are equivalent.*

Proof. The tableau before the pivot operation

$$
\begin{array}{c|cc|c}
 & y_l & y_j & \\
\hline
x_k & a^* & b & = -y_{n+k} \\
x_i & c & d & = -y_{n+i} \\
\hline
 & \shortparallel & \shortparallel & \\
 & x_{m+l} & x_{m+j} &
\end{array}
$$

is equivalent to the system of equations

$$
\begin{aligned}
& \left\{\begin{array}{l} ax_k + cx_i = x_{m+l} \\ bx_k + dx_i = x_{m+j} \end{array}\right. \quad \text{and} \quad \left\{\begin{array}{l} ay_l + by_j = -y_{n+k} \\ cy_l + dy_j = -y_{n+i} \end{array}\right. \\
\Leftrightarrow \ & \left\{\begin{array}{l} -x_{m+l} + cx_i = -ax_k \\ bx_k + dx_i = x_{m+j} \end{array}\right. \quad \text{and} \quad \left\{\begin{array}{l} y_{n+k} + by_j = -ay_l \\ cy_l + dy_j = -y_{n+i} \end{array}\right. \\
\Leftrightarrow \ & \left\{\begin{array}{l} \frac{1}{a}x_{m+l} - \frac{c}{a}x_i = x_k \\ bx_k + dx_i = x_{m+j} \end{array}\right. \quad \text{and} \quad \left\{\begin{array}{l} \frac{1}{a}y_{n+k} + \frac{b}{a}y_j = -y_l \\ cy_l + dy_j = -y_{n+i} \end{array}\right. \\
\Leftrightarrow \ & \left\{\begin{array}{l} \frac{1}{a}x_{m+l} - \frac{c}{a}x_i = x_k \\ b\left(\frac{1}{a}x_{m+l} - \frac{c}{a}x_i\right) + dx_i = x_{m+j} \end{array}\right. \\
& \qquad\qquad \text{and} \left\{\begin{array}{l} \frac{1}{a}y_{n+k} + \frac{b}{a}y_j = -y_l \\ c\left(\frac{1}{a}y_{n+k} + \frac{b}{a}y_j\right) + dy_j = -y_{n+i} \end{array}\right. \\
\Leftrightarrow \ & \left\{\begin{array}{l} \frac{1}{a}x_{m+l} - \frac{c}{a}x_i = x_k \\ \frac{b}{a}x_{m+l} + \left(d - \frac{bc}{a}\right) x_i = x_{m+j} \end{array}\right. \\
& \qquad\qquad \text{and} \left\{\begin{array}{l} \frac{1}{a}y_{n+k} + \frac{b}{a}y_j = -y_l \\ -\frac{c}{a}y_{n+k} + \left(d - \frac{bc}{a}\right) y_j = -y_{n+i} \end{array}\right.
\end{aligned}
$$

which is equivalent to the tableau

$$
\begin{array}{c|cc|c}
 & y_{n+k} & y_j & \\
\hline
x_{m+l} & \frac{1}{a} & \frac{b}{a} & = -y_l \\
x_i & -\frac{c}{a} & d - \frac{bc}{a} & = -y_{n+i} \\
\hline
 & \shortparallel & \shortparallel & \\
 & x_k & x_{m+j} &
\end{array}
$$

The above calculation shows that the constraints before and after a pivot operation are equivalent, and the values of the objective functions f and g for any given $x_1, \cdots, x_m, x_{m+1}, \cdots, x_{m+n}$ and $y_1, \cdots, y_n, y_{n+1}, \cdots, y_{n+m}$ satisfying the constraints remain unchanged. □

For each pair of basic forms, there associates a pair of basic solutions which will be defined below. Note that the basic solutions are not really solutions to the primal and dual problems because basic solutions are not necessarily feasible.

Definition 2.3.3 (Basic solution)**.** *Suppose we have a pair of basic forms represented by the tableau*

$$
\begin{array}{c|ccc|c|l}
 & y_1 & \cdots & y_n & -1 & \\
\hline
x_1 & a_{11} & \cdots & a_{1n} & b_1 & = -y_{n+1} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \quad\vdots \\
x_m & a_{m1} & \cdots & a_{mn} & b_m & = -y_{n+m} \\
\hline
-1 & c_1 & \cdots & c_n & -d & = f \\
 & \shortparallel & \cdots & \shortparallel & \shortparallel & \\
 & x_{m+1} & \cdots & x_{m+n} & g &
\end{array}
$$

The **basic solution** *to the basic form is*

$$
\begin{aligned}
&x_1 = x_2 = \cdots = x_m = 0, x_{m+1} = -c_1, x_{m+2} = -c_2, \cdots, x_{m+n} = -c_n \\
&y_1 = y_2 = \cdots = y_n = 0, y_{n+1} = b_1, y_{n+2} = b_2, \cdots, y_{n+m} = b_m
\end{aligned}
$$

The basic solutions are obtained by setting the independent variables, that is the variables at the top and at the left, to be 0 and then solving for the basic variables, that is the variables at the bottom and at the right, by the constraints.

The basic solutions always satisfy the equalities in the constraints, but they may not be feasible since some variables may have negative values. However if both the dual and primal basic solutions are feasible, then they must be optimal.

Theorem 2.3.4. *Suppose we have a pair of basic forms.*

1. *The basic solution to the primal basic form is feasible if and only if* $b_1, b_2, \cdots, b_m \geq 0$.

2. *The basic solution to the dual basic form is feasible if and only if* $c_1, c_2, \cdots, c_n \le 0$.

3. *The pair of basic solutions are optimal if* $b_1, \cdots, b_m \ge 0$ *and* $c_1, \cdots, c_n \le 0$.

Proof. Observe that the basic solutions always satisfy the equalities $\mathbf{x}A - \mathbf{c} = (x_{m+1}, \cdots, x_{m+n})$ and $A\mathbf{y}^T - \mathbf{b}^T = -(y_{n+1}, \cdots, y_{n+m})^T$ of the constraints.

1. The basic solution to the primal basic form is $(y_1, \cdots, y_n, y_{n+1}, \cdots, y_{n+m}) = (0, \cdots, 0, b_1, \cdots, b_m)$. Thus it is feasible if and only if all the variables are non-negative which is equivalent to $b_1, b_2, \cdots, b_m \ge 0$.

2. The basic solution to the dual basic form is $(x_1, \cdots, x_m, x_{m+1}, \cdots, x_{m+n}) = (0, \cdots, 0, -c_1, \cdots, -c_n)$. Thus it is is feasible if and only if all the variables are non-negative which is equivalent to $c_1, c_2, \cdots, c_n \le 0$.

3. Suppose $b_1, b_2, \cdots, b_m \ge 0$ and $c_1, c_2, \cdots, c_n \le 0$. For any feasible vectors $(x_1, \cdots, x_m, x_{m+1}, \cdots, x_{m+n})$ of the dual basic form and $(y_1, \cdots, y_n, y_{n+1}, \cdots, y_{n+m})$ of the primal basic form, we have

$$\begin{aligned} f(y_1, \cdots, y_n) &= (c_1, \cdots, c_n)(y_1, \cdots, y_n)^T + d \\ &\le (x_1, \cdots, x_m)A(y_1, \cdots, y_n)^T + d \\ &\le (x_1, \cdots, x_m)(b_1, \cdots, b_m)^T + d \\ &= g(x_1, \cdots, x_m) \end{aligned}$$

   On the other hand, the basic solutions $(x_1, \cdots, x_m, x_{m+1}, \cdots, x_{m+n}) = (0, \cdots, 0, -c_1, \cdots, -c_n)$ and $(y_1, \cdots, y_n, y_{n+1}, \cdots, y_{n+m}) = (0, \cdots, 0, b_1, \cdots, b_m)$ are feasible and

$$f(0, \cdots, 0) = d = g(0, \cdots, 0)$$

   Therefore f attains its maximin and g attains its minimum at the basic solutions.

□

In practice, we do not write down the basic variables. We would swap the variables at the left and at the top when preforming pivot operation. One may find the basic and independent variables by referring to the following table.

	Left	Top
x_i	x_i is independent variable y_{n+i} is basic variable	x_i is basic variable y_{n+i} is independent variable
y_j	y_j is basic variable x_{m+j} is independent variable	y_j is independent variable x_{m+j} is basic variable

In other words, when we write down a tableau of the form

$$\begin{array}{c|cc|c} & x_i & y_l & -1 \\ \hline y_j & \multicolumn{2}{c|}{\multirow{2}{*}{A}} & b_i \\ x_k & & & b_k \\ \hline -1 & c_j & c_l & -d \end{array}$$

the basic solution associated with it is

$$\begin{array}{l} x_i = -c_j, x_k = 0, x_{m+j} = 0, x_{m+l} = -c_l \\ y_j = b_i, y_l = 0, y_{n+i} = 0, y_{n+k} = b_k \end{array}$$

and the genuine tableau is

$$\begin{array}{c|cc|l} & y_{n+i} & y_l & -1 \\ \hline x_{m+j} & \multicolumn{2}{c|}{\multirow{2}{*}{A}} & b_i = -y_j \\ x_k & & & b_k = -y_{n+k} \\ \hline -1 & c_j & c_l & -d \\ & \shortparallel & \shortparallel & \\ & x_i & x_{m+l} & \end{array}$$

Example 2.3.5. *Solve the following primal problem.*

$$\begin{array}{rl} \max & f = 6y_1 + 4y_2 + 5y_3 + 150 \\ \textit{subject to} & 2y_1 + y_2 + y_3 \leq 180 \\ & y_1 + 2y_2 + 3y_3 \leq 300 \\ & 2y_1 + 2y_2 + y_3 \leq 240 \end{array}$$

Solution. Set up the tableau and perform pivot operations successively. The pivoting entries are marked with asterisks.

	y_1	y_2	y_3	-1
x_1	2^*	1	1	180
x_2	1	2	3	300
x_3	2	2	1	240
-1	6	4	5	-150

$\longrightarrow$

	x_1	y_2	y_3	-1
y_1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	90
x_2	$-\frac{1}{2}$	$\frac{3}{2}$	$\frac{5}{2}$	210
x_3	-1	1^*	0	60
-1	-3	1	2	-690

$\longrightarrow$

	x_1	x_3	y_3	-1
y_1	1	$-\frac{1}{2}$	$\frac{1}{2}$	60
x_2	1	$-\frac{3}{2}$	$\frac{5}{2}^*$	120
y_2	-1	1	0	60
-1	-2	-1	2	-750

$\longrightarrow$

	x_1	x_3	x_2	-1
y_1	$\frac{4}{5}$	$-\frac{1}{5}$	$-\frac{1}{5}$	36
y_3	$\frac{2}{5}$	$-\frac{3}{5}$	$\frac{2}{5}$	48
y_2	-1	1^*	0	60
-1	$-\frac{14}{5}$	$\frac{1}{5}$	$-\frac{4}{5}$	-846

$\longrightarrow$

	x_1	y_2	x_2	-1
y_1	$\frac{3}{5}$	$\frac{1}{5}$	$-\frac{1}{5}$	48
y_3	$-\frac{1}{5}$	$\frac{3}{5}$	$\frac{2}{5}$	84
x_3	-1	1	0	60
-1	$-\frac{13}{5}$	$-\frac{1}{5}$	$-\frac{4}{5}$	-858

The independent variables are y_2, y_4, y_5 and the basic variables are y_1, y_3, y_6. The basic solution is

$$y_2 = y_4 = y_5 = 0, y_1 = 48, y_3 = 84, y_6 = 60$$

Thus an optimal vector for the primal problem is

$$(y_1, y_2, y_3) = (48, 0, 84)$$

The maximum value of f is 858.

We may also write down an optimal solution to the dual problem. The dual problem is

$$\begin{array}{rl} \min & g = 180x_1 + 300x_2 + 240x_3 + 150 \\ \text{subject to} & 2x_1 + x_2 + 2x_3 \geq 6 \\ & x_1 + 2x_2 + 2x_3 \geq 4 \\ & x_1 + 3x_2 + x_3 \geq 5 \end{array}$$

From the last tableau, the independent variables are x_3, x_4, x_6 and the basic variables are x_1, x_2, x_5. The basic solution is

$$x_3 = x_4 = x_6 = 0, x_1 = \frac{13}{5}, x_2 = \frac{4}{5}, x_5 = \frac{1}{5}$$

Therefore an optimal vector for the dual problem is

$$(x_1, x_2, x_3) = \left(\frac{13}{5}, \frac{4}{5}, 0\right)$$

The minimum value of g is 858 which is equal to the maximum value of f.□

To use simplex method solving a game matrix, first we add a constant k to every entry so that the entries are all non-negative and there is no zero column. This is done to make sure that the value of the new matrix is positive. Then we take $\mathbf{b} = (1, \cdots, 1) \in \mathbb{R}^m$, $\mathbf{c} = (1, \cdots, 1) \in \mathbb{R}^n$ to set up the initial tableau

$$\begin{array}{c|ccc|c} & y_1 & \cdots & y_n & \\ \hline x_1 & a_{11} & \cdots & a_{1n} & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_m & a_{m1} & \cdots & a_{mn} & 1 \\ \hline & 1 & \cdots & 1 & 0 \end{array}$$

and apply the simplex algorithm. Then the value of the game matrix is

$$\nu = \frac{1}{d} - k$$

where d is the maximum value of f or the minimum value of g, and k is the constant which is added to the game matrix at the beginning. A maximin strategy for the row player is

$$\mathbf{p} = \frac{1}{d}\mathbf{x} = \frac{1}{d}(x_1, x_2, \cdots, x_m)$$

and a minimax strategy for the column player is

$$\mathbf{q} = \frac{1}{d}\mathbf{y} = \frac{1}{d}(y_1, x_2, \cdots, y_n)$$

To avoid making mistakes, one may check that the following conditions must be satisfied in every step.

1. The rightmost number in each row is always non-negative. This is guaranteed by the choice of the pivoting entry.

2. The value of the number in the lower right corner is always equal to the sum of those entries in the lower row which associate with x_i's at the top row (and similarly equal to the sum of those entries at the rightmost column associate with y_j's at the leftmost column.)

3. The value of the number in the lower right corner never increases.

Finally, one may also check that the result should satisfy the following two conditions.

1. Every entry of $\mathbf{p}A$ is larger than or equal to ν.

2. Every entry of $A\mathbf{q}^T$ is less than or equal to ν.

Example 2.3.6. *Solve the two-person zero sum game with game matrix*

$$\begin{pmatrix} -1 & 5 & 3 & 2 \\ 6 & -1 & 0 & 4 \end{pmatrix}$$

Solution. Add $k = 1$ to each of the entries, we obtain the matrix

$$\begin{pmatrix} 0 & 6 & 4 & 3 \\ 7 & 0 & 1 & 5 \end{pmatrix}$$

Applying simplex algorithm, we have

	y_1	y_2	y_3	y_4	-1
x_1	0	6	4	3	1
x_2	7^*	0	1	5	1
-1	1	1	1	1	0

$\longrightarrow$

	x_2	y_2	y_3	y_4	-1
x_1	0	6^*	4	3	1
y_1	$\frac{1}{7}$	0	$\frac{1}{7}$	$\frac{5}{7}$	$\frac{1}{7}$
-1	$-\frac{1}{7}$	1	$\frac{6}{7}$	$\frac{2}{7}$	$-\frac{1}{7}$

$\longrightarrow$

	x_2	x_1	y_3	y_4	-1
y_2	0	$\frac{1}{6}$	$\frac{2}{3}^*$	$\frac{1}{2}$	$\frac{1}{6}$
y_1	$\frac{1}{7}$	0	$\frac{1}{7}$	$\frac{5}{7}$	$\frac{1}{7}$
-1	$-\frac{1}{7}$	$-\frac{1}{6}$	$\frac{4}{21}$	$-\frac{3}{14}$	$-\frac{13}{42}$

$\longrightarrow$

	x_2	x_1	y_2	y_4	-1
y_3	0	$\frac{1}{4}$	$\frac{3}{2}$	$\frac{3}{4}$	$\frac{1}{4}$
y_1	$-\frac{1}{7}$	$-\frac{1}{28}$	$-\frac{3}{14}$	$\frac{17}{28}$	$\frac{3}{28}$
-1	$-\frac{1}{7}$	$-\frac{3}{14}$	$-\frac{2}{7}$	$-\frac{5}{14}$	$-\frac{5}{14}$

The independent variables are $x_3, x_5, y_2, y_4, y_5, y_6$ and the basic variables are $x_1, x_2, x_4, x_6, y_1, y_3$. The basic solution is

$$x_3 = x_5 = 0, x_1 = \frac{3}{14}, x_2 = \frac{1}{7}, x_4 = \frac{2}{7}, x_6 = \frac{5}{14}$$
$$y_2 = y_4 = y_5 = y_6 = 0, y_1 = \frac{3}{28}, y_3 = \frac{1}{4}$$

The optimal value is $d = \dfrac{5}{14}$. Therefore a maximin strategy for the row player is

$$\mathbf{p} = \frac{1}{d}(x_1, x_2) = \frac{14}{5}\left(\frac{3}{14}, \frac{1}{7}\right) = \left(\frac{3}{5}, \frac{2}{5}\right)$$

A minimax strategy for the column player is

$$\mathbf{q} = \frac{1}{d}(y_1, y_2, y_3, y_4) = \frac{14}{5}\left(\frac{3}{28}, 0, \frac{1}{4}, 0\right) = \left(\frac{3}{10}, 0, \frac{7}{10}, 0\right)$$

The value of the game is

$$\nu = \frac{1}{d} - k = \frac{14}{5} - 1 = \frac{9}{5}$$

□

Example 2.3.7. *Solve the two-person zero sum game with game matrix*

$$A = \begin{pmatrix} 2 & -1 & 6 \\ 0 & 1 & -1 \\ -2 & 2 & 1 \end{pmatrix}$$

Solution. Add 2 to each of the entries, we obtain the matrix

$$\begin{pmatrix} 4 & 1 & 8 \\ 2 & 3 & 1 \\ 0 & 4 & 3 \end{pmatrix}$$

Applying simplex method, we have

$$\begin{array}{c|ccc|c} & y_1 & y_2 & y_3 & -1 \\ \hline x_1 & 4^* & 1 & 8 & 1 \\ x_2 & 2 & 3 & 1 & 1 \\ x_3 & 0 & 4 & 3 & 1 \\ \hline -1 & 1 & 1 & 1 & 0 \end{array} \longrightarrow \begin{array}{c|ccc|c} & x_1 & y_2 & y_3 & -1 \\ \hline y_1 & \frac{1}{4} & \frac{1}{4} & 2 & \frac{1}{4} \\ x_2 & -\frac{1}{2} & \frac{5}{2}^* & -3 & \frac{1}{2} \\ x_3 & 0 & 4 & 3 & 1 \\ \hline -1 & -\frac{1}{4} & \frac{3}{4} & -1 & -\frac{1}{4} \end{array}$$

$$\longrightarrow \begin{array}{c|ccc|c} & x_1 & x_2 & y_3 & -1 \\ \hline y_1 & \frac{3}{10} & -\frac{1}{10} & \frac{23}{10} & \frac{1}{5} \\ y_2 & -\frac{1}{5} & \frac{2}{5} & -\frac{6}{5} & \frac{1}{5} \\ x_3 & \frac{4}{5} & -\frac{8}{5} & \frac{39}{5} & \frac{1}{5} \\ \hline -1 & -\frac{1}{10} & -\frac{3}{10} & -\frac{1}{10} & -\frac{2}{5} \end{array}$$

The independent variables are $x_3, x_4, x_5, y_3, y_4, y_5$ and the basic variables are $x_1, x_2, x_6, y_1, y_2, y_6$. The basic solution is

$$\begin{aligned} &x_3 = x_4 = x_5 = 0, x_1 = \frac{1}{10}, x_2 = \frac{3}{10}, x_6 = \frac{1}{10} \\ &y_3 = y_4 = y_5 = 0, y_1 = \frac{1}{5}, y_2 = \frac{1}{5}, y_6 = \frac{1}{5} \end{aligned}$$

The optimal value is $d = \frac{2}{5}$. Therefore a maximin strategy for the row player is

$$\mathbf{p} = \frac{1}{d}(x_1, x_2, x_3) = \frac{5}{2}\left(\frac{1}{10}, \frac{3}{10}, 0\right) = \left(\frac{1}{4}, \frac{3}{4}, 0\right)$$

A minimax strategy for the column player is

$$\mathbf{q} = \frac{1}{d}(y_1, y_2, y_3) = \frac{5}{2}\left(\frac{1}{5}, \frac{1}{5}, 0\right) = \left(\frac{1}{2}, \frac{1}{2}, 0\right)$$

The value of the game is

$$\nu = \frac{1}{d} - k = \frac{5}{2} - 1 = \frac{1}{2}$$

One may check the result by the following calculations

$$\begin{aligned} \mathbf{p}A &= \left(\frac{1}{4}, \frac{3}{4}, 0\right)\begin{pmatrix} 2 & -1 & 6 \\ 0 & 1 & -1 \\ -2 & 2 & 1 \end{pmatrix} = \left(\frac{1}{2}, \frac{1}{2}, \frac{3}{4}\right) \\ A\mathbf{q}^T &= \begin{pmatrix} 2 & -1 & 6 \\ 0 & 1 & -1 \\ -2 & 2 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix} \end{aligned}$$

One sees that the row player may guarantee that his payoff is at least $\frac{1}{2}$ by using $\mathbf{p} = \left(\frac{1}{4}, \frac{3}{4}, 0\right)$ and the column player may guarantee that the payoff to the row player is at most $\frac{1}{2}$ by using $\mathbf{q} = \left(\frac{1}{2}, \frac{1}{2}, 0\right)$. $\square$

2.4 Minimax theorem

In this section, we prove the minimax theorem (Theorem 1.1.10). The theorem was first published by John von Neumann in 1928. Another way to state the minimax theorem is that the row value and the column value of a matrix are always the same.

Definition 2.4.1 (Row and column values)**.** *Let A be an $m \times n$ matrix.*

1. *The* **row value** *of A is defined*[1] *by*

$$\nu_r(A) = \max_{\mathbf{x}\in\mathcal{P}^m} \min_{\mathbf{y}\in\mathcal{P}^n} \mathbf{x}A\mathbf{y}^T$$

2. *The* **column value** *of A is defined by*

$$\nu_c(A) = \min_{\mathbf{y}\in\mathcal{P}^n} \max_{\mathbf{x}\in\mathcal{P}^m} \mathbf{x}A\mathbf{y}^T$$

The row value $\nu_r(A)$ of a game matrix A is the largest payoff of the row player that he may guarantee himself. The column value $\nu_c(A)$ of A is the least payoff that the column player may guarantee that the row player cannot surpass. The strategies for the players to achieve these goals are called maximin and minimax strategies.

Definition 2.4.2 (Maximin and minimax strategies)**.** *Let A be an $m \times n$ matrix.*

1. *A* **maximin strategy** *is a strategy $\mathbf{p} \in \mathcal{P}^m$ for the row player such that*
$$\min_{\mathbf{y}\in\mathcal{P}^n} \mathbf{p}A\mathbf{y}^T = \max_{\mathbf{x}\in\mathcal{P}^m} \min_{\mathbf{y}\in\mathcal{P}^n} \mathbf{x}A\mathbf{y}^T = \nu_r(A)$$

2. *A* **minimax strategy** *is a strategy $\mathbf{q} \in \mathcal{P}^n$ for the column player such that*
$$\max_{\mathbf{x}\in\mathcal{P}^m} \mathbf{x}A\mathbf{q}^T = \min_{\mathbf{y}\in\mathcal{P}^n} \max_{\mathbf{x}\in\mathcal{P}^m} \mathbf{x}A\mathbf{y}^T = \nu_c(A)$$

It can be seen readily that we always have $\nu_r(A) \le \nu_c(A)$ for any matrix A and we give a rigorous proof here.

[1] Note that since the payoff function $\pi(\mathbf{x}, \mathbf{y}) = \mathbf{x}A\mathbf{y}^T$ is continuous and the sets $\mathcal{P}^m, \mathcal{P}^n$ are compact, that is closed and bounded, the payoff function attains its maximum and minimum by extreme value theorem.

Theorem 2.4.3. *For any $m \times n$ matrix A, we have*

$$\nu_r(A) \leq \nu_c(A)$$

Proof. Let $\mathbf{p} \in \mathcal{P}^m$ be a maximin strategy for the row player and $\mathbf{q} \in \mathcal{P}^n$ be a minimax strategy for the column player. Then we have

$$\begin{aligned}
\nu_r(A) &= \max_{\mathbf{x}\in\mathcal{P}^m} \min_{\mathbf{y}\in\mathcal{P}^n} \mathbf{x}A\mathbf{y}^T \\
&= \min_{\mathbf{y}\in\mathcal{P}^n} \mathbf{p}A\mathbf{y}^T \\
&\leq \mathbf{p}A\mathbf{q}^T \\
&\leq \max_{\mathbf{x}\in\mathcal{P}^m} \mathbf{x}A\mathbf{q}^T \\
&= \min_{\mathbf{y}\in\mathcal{P}^n} \max_{\mathbf{x}\in\mathcal{P}^m} \mathbf{x}A\mathbf{y}^T \\
&= \nu_c(A)
\end{aligned}$$

□

Before we prove the minimax theorem, let's study some properties of convex sets.

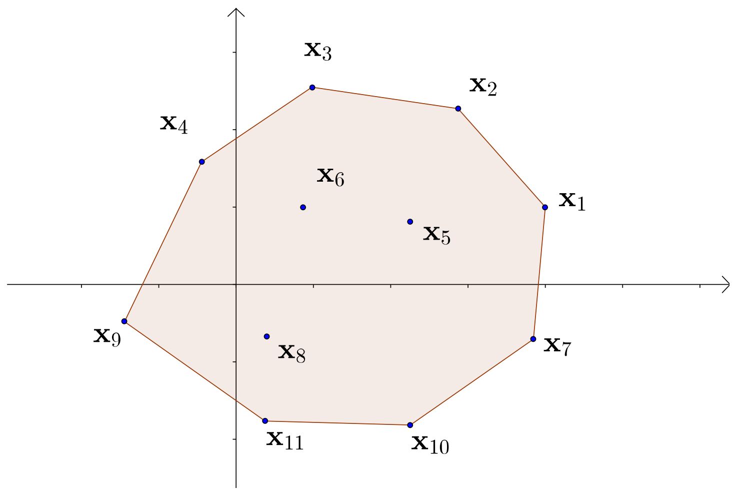
Definition 2.4.4 (Convex set). *A set $C \subset \mathbb{R}^n$ is said to be* **convex** *if*

$$\lambda\mathbf{x} + (1-\lambda)\mathbf{y} \in C \text{ for any } \mathbf{x}, \mathbf{y} \in C,\ 0 \leq \lambda \leq 1$$

Geometrically, a set $C \subset \mathbb{R}^n$ is convex if the line segment joining any two points in C is contained in C. It is easy to see from the definition that intersection of convex sets is convex.

Definition 2.4.5 (Convex hull). *The* **convex hull** *of a set $\{\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_k\}$ of vectors in $\mathbb{R}^n$ is defined by*

$$\begin{aligned}
&\mathrm{Conv}(\{\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_k\}) \\
= \ &\{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} = \sum_{i=1}^{k} \lambda_i \mathbf{x}_i \text{ with } \lambda_i \geq 0 \text{ for all } i \text{ and } \sum_{i=1}^{k} \lambda_i = 1\}
\end{aligned}$$



The convex hull of a set of vectors can also be defined as the smallest convex set which contains all vectors in the set.

To prove the minimax theorem, we prove a lemma concerning properties of convex sets. Recall that the standard inner product and the norm on $\mathbb{R}^n$ are defined as follows. For any $\mathbf{x} = (x_1, x_2, \cdots, x_n), \mathbf{y} = (y_1, y_2, \cdots, y_n) \in \mathbb{R}^n$,

1. $\langle \mathbf{x}, \mathbf{y} \rangle = x_1y_1 + x_2y_2 + \cdots + x_ny_n$
2. $\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle} = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$

The following lemma says that we can always use a plane to separate the origin and a closed convex set C not containing the origin. It is a special case of the **hyperplane separation theorem**[2].

Lemma 2.4.6. *Let $C \subset \mathbb{R}^n$ be a closed convex set with $\mathbf{0} \notin C$. Then there exists $\mathbf{z} \in C$ such that*

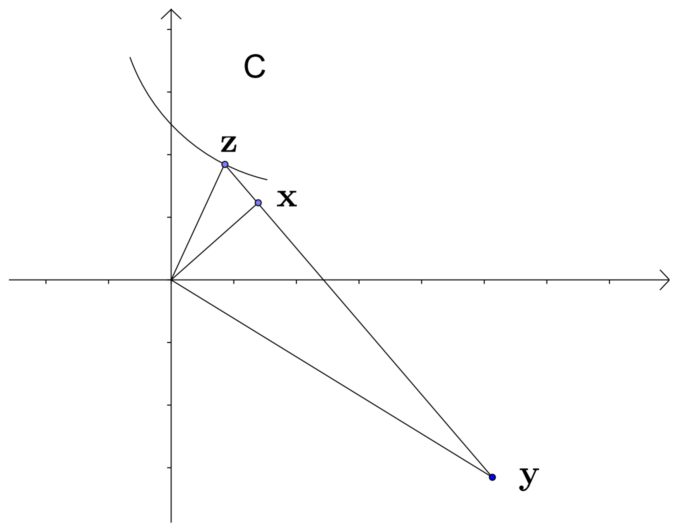
$$\langle \mathbf{z}, \mathbf{y} \rangle > 0 \text{ for any } \mathbf{y} \in C$$

[2]The hyperplane separation theorem says that we can always use a hyperplane to separate two given sets which are closed and convex, and at least one of them is bounded.

Proof. Since C is closed, there exists $\mathbf{z} \in C$ such that

$$\|\mathbf{z}\| = \min_{\mathbf{y} \in C} \|\mathbf{y}\|$$

We are going to prove that $\langle \mathbf{z}, \mathbf{y} \rangle > 0$ for any $\mathbf{y} \in C$ by contradiction. Suppose there exists $\mathbf{y} \in C$ such that $\langle \mathbf{z}, \mathbf{y} \rangle \leq 0$. Let $\mathbf{x} \in \mathbb{R}^n$ be a point which lies on the straight line passing through $\mathbf{z}$, $\mathbf{y}$, and is orthogonal to $\mathbf{z}-\mathbf{y}$. The point $\mathbf{x}$ lies on the line segment joining $\mathbf{z}$, $\mathbf{y}$, that is lying between $\mathbf{z}$ and $\mathbf{y}$, because $\langle \mathbf{z}, \mathbf{y} \rangle \leq 0$.



Since $\mathbf{z}, \mathbf{y} \in C$ and C is convex, we have $\mathbf{x} \in C$. (The expression for $\mathbf{x}$ is not important in the proof but let's include here for reference

$$\mathbf{x} = \frac{\langle \mathbf{y} - \mathbf{z}, \mathbf{y} \rangle}{\|\mathbf{y} - \mathbf{z}\|^2} \mathbf{z} + \frac{\langle \mathbf{z} - \mathbf{y}, \mathbf{z} \rangle}{\|\mathbf{y} - \mathbf{z}\|^2} \mathbf{y}$$

Note that $\frac{\langle \mathbf{y}-\mathbf{z}, \mathbf{y} \rangle}{\|\mathbf{y}-\mathbf{z}\|^2}, \frac{\langle \mathbf{z}-\mathbf{y}, \mathbf{z} \rangle}{\|\mathbf{y}-\mathbf{z}\|^2} \geq 0$ because $\langle \mathbf{z}, \mathbf{y} \rangle \leq 0$ and $\frac{\langle \mathbf{y}-\mathbf{z}, \mathbf{y} \rangle}{\|\mathbf{y}-\mathbf{z}\|^2} + \frac{\langle \mathbf{z}-\mathbf{y}, \mathbf{z} \rangle}{\|\mathbf{y}-\mathbf{z}\|^2} = 1$ which shows that $\mathbf{x}$ lies on the line segment joining $\mathbf{z}$, $\mathbf{y}$.)

Moreover, we have

$$\begin{aligned} \|\mathbf{z}\|^2 &= \|\mathbf{x} + (\mathbf{z} - \mathbf{x})\|^2 \\ &= \|\mathbf{x}\|^2 + \|(\mathbf{z} - \mathbf{x})\|^2 \text{ (since } \mathbf{x} \perp \mathbf{z} - \mathbf{x}) \\ &> \|\mathbf{x}\|^2 \end{aligned}$$

which contradicts that $\mathbf{z}$ is a point in C closest to the origin $\mathbf{0}$. □

The following theorem says that for any matrix A, we have either $\nu_r(A) > 0$ or $\nu_c(A) \le 0$. The key of the proof is to consider the convex hull C generated by the column vectors of A and the standard basis for $\mathbb{R}^m$, and study the two cases $\mathbf{0} \notin C$ and $\mathbf{0} \in C$.

Theorem 2.4.7. *Let A be an $m \times n$ matrix. Then one of the following statements holds.*

1. *There exists probability vector $\mathbf{x} \in \mathcal{P}^m$ such that $\mathbf{x}A > \mathbf{0}$, that is all coordinates of $\mathbf{x}A$ are positive. In this case, $\nu_r(A) > 0$.*
2. *There exists probability vector $\mathbf{y} \in \mathcal{P}^n$ such that $A\mathbf{y}^T \le \mathbf{0}$, that is all coordinates of $A\mathbf{y}^T$ are non-positive. In this case, $\nu_c(A) \le 0$.*

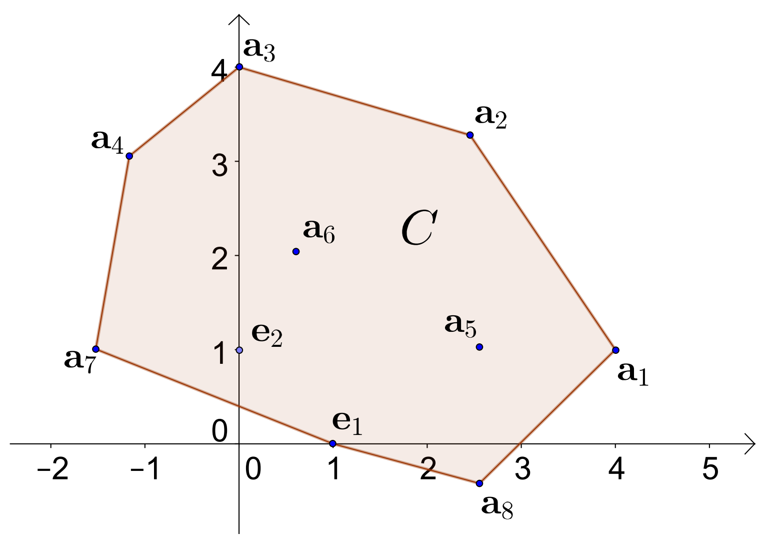
Proof. For $j = 1, 2, \cdots, n$, let

$$\mathbf{a}_j = (a_{1j}, a_{2j}, \cdots, a_{mj}) \in \mathbb{R}^m$$

In other words, $\mathbf{a}_1^T, \mathbf{a}_2^T, \cdots, \mathbf{a}_n^T$ are the column vectors of A and we may write $A = [\mathbf{a}_1^T, \mathbf{a}_2^T, \cdots, \mathbf{a}_n^T]$. Let

$$C = \mathrm{Conv}(\{\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n, \mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_m\})$$

be the convex hull of $\{\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n, \mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_m\}$ where $\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_m\}$ is the standard basis for $\mathbb{R}^m$.



We are going to prove that the two statements in the theorem correspond to the two cases $\mathbf{0} \notin C$ and $\mathbf{0} \in C$.

Case 1. Suppose $\mathbf{0} \notin C$. Then by Lemma 2.4.6, there exists $\mathbf{z} = (z_1, z_2, \cdots, z_m) \in \mathbb{R}^m$ such that

$$\langle \mathbf{z}, \mathbf{y} \rangle > 0 \text{ for any } \mathbf{y} \in C$$

In particular, we have

$$\langle \mathbf{z}, \mathbf{e}_i \rangle = z_i > 0 \text{ for any } i = 1, 2, \cdots, m$$

Then we may take

$$\mathbf{x} = \frac{\mathbf{z}}{z_1 + z_2 + \cdots + z_m} \in \mathcal{P}^m$$

and we have

$$\langle \mathbf{x}, \mathbf{a}_j \rangle = \frac{\langle \mathbf{z}, \mathbf{a}_j \rangle}{z_1 + z_2 + \cdots + z_m} > 0 \text{ for any } j = 1, 2, \cdots, n$$

which means $\mathbf{x}A > \mathbf{0}$. Let $\alpha > 0$ be the smallest coordinate of the vector $\mathbf{x}A$ and we have

$$\nu_r(A) \geq \min_{\mathbf{y} \in \mathcal{P}^n} \mathbf{x}A\mathbf{y}^T \geq \alpha > 0$$

Case 2. Suppose $\mathbf{0} \in C$. Then there exists $\lambda_1, \lambda_2, \cdots, \lambda_{m+n}$ with $\lambda_i \geq 0$ for all i, and $\lambda_1 + \lambda_2 + \cdots + \lambda_{m+n} = 1$ such that

$$\lambda_1 \mathbf{a}_1 + \lambda_2 \mathbf{a}_2 + \cdots + \lambda_n \mathbf{a}_n + \lambda_{n+1} \mathbf{e}_1 + \lambda_{n+2} \mathbf{e}_2 + \cdots + \lambda_{n+m} \mathbf{e}_m = \mathbf{0}$$

which implies

$$\begin{aligned} & A\left(\lambda_1, \lambda_2, \cdots, \lambda_n\right)^T \\ = \; & \lambda_1 \mathbf{a}_1^T + \lambda_2 \mathbf{a}_2^T + \cdots + \lambda_n \mathbf{a}_n^T \\ = \; & -(\lambda_{n+1} \mathbf{e}_1^T + \lambda_{n+1} \mathbf{e}_2^T + \cdots + \lambda_{n+m} \mathbf{e}_m^T) \\ = \; & -\left(\lambda_{n+1}, \lambda_{n+2}, \cdots, \lambda_{n+m}\right)^T \end{aligned}$$

Since $\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_m\}$ are linearly independent, at least one of $\lambda_1, \lambda_2, \cdots, \lambda_n$ is positive for otherwise all $\lambda_1, \lambda_2, \cdots, \lambda_{m+n}$ are zero which contradicts $\lambda_1 + \lambda_2 + \cdots + \lambda_{m+n} = 1$. Then we may take

$$\mathbf{y} = \frac{(\lambda_1, \lambda_2, \cdots, \lambda_n)}{\lambda_1 + \lambda_2 + \cdots + \lambda_n} \in \mathcal{P}^n$$

and we have

$$Ay^T = -\frac{1}{\lambda_1 + \lambda_2 + \cdots + \lambda_n} \begin{pmatrix} \lambda_{n+1} \\ \vdots \\ \lambda_{n+m} \end{pmatrix} \leq \mathbf{0}$$

which implies

$$v_c(A) \leq \max_{\mathbf{x} \in \mathcal{P}^m} \mathbf{x} A \mathbf{y}^T \leq 0$$

□

Now we give the proof of the minimax theorem (Theorem 1.1.10) which can be stated in the following form.

Theorem 2.4.8 (Minimax theorem)**.** *For any matrix A, the row value and columns value of A are equal. In other words, we have*

$$\nu_r(A) = \nu_c(A)$$

Proof. It has been proved that $\nu_r(A) \leq \nu_c(A)$ for any matrix A (Theorem 2.4.3). We are going to prove that $\nu_c(A) \leq \nu_r(A)$ by contradiction. Suppose there exists matrix A such that $\nu_r(A) < \nu_c(A)$. Let k be a real number such that $\nu_r(A) < k < \nu_c(A)$. Let A' be the matrix obtained by subtracting every entry of A by k. Then $\nu_r(A') = \nu_r(A) - k < 0$ and $\nu_c(A') = \nu_c(A) - k > 0$ which is impossible by applying Theorem 2.4.7 to A'. The contradiction shows that $\nu_c(A) \leq \nu_r(A)$ for any matrix A and the proof of the minimax theorem is complete. □

Exercise 2

1. Solve the following primal problems. Then write down the dual problems and the solutions to the dual problems.

   (a)
$$\begin{array}{rl} \max & f = 3y_1 + 5y_2 + 4y_3 + 12 \\ \text{subject to} & 3y_1 + 2y_2 + 2y_3 \leq 15 \\ & 4y_2 + 5y_3 \leq 24 \end{array}$$

   (b)
$$\begin{array}{rl} \max & f = 2y_1 + 4y_2 + 3y_3 + y_4 \\ \text{subject to} & 3y_1 + y_2 + y_3 + 4y_4 \leq 12 \\ & y_1 - 3y_2 + 2y_3 + 3y_4 \leq 7 \\ & 2y_1 + y_2 + 3y_3 - y_4 \leq 10 \end{array}$$

2. Solve the zero sum games with the following game matrices, that is find the value of the game, a maximin strategy for the row player and a minimax strategy for the column player.

   (a) $\begin{pmatrix} 2 & -3 & 3 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{pmatrix}$

   (b) $\begin{pmatrix} 3 & 1 & -5 \\ -1 & -2 & 6 \\ -2 & -1 & 3 \end{pmatrix}$

   (c) $\begin{pmatrix} 3 & 0 & 1 \\ -1 & 2 & -2 \\ 0 & 1 & -1 \end{pmatrix}$

   (d) $\begin{pmatrix} 2 & 0 & -2 \\ -1 & -3 & 3 \\ -2 & 2 & 0 \end{pmatrix}$

   (e) $\begin{pmatrix} 1 & -1 & 1 \\ -2 & 0 & -1 \\ 1 & -2 & 2 \\ -1 & 1 & -2 \end{pmatrix}$

   (f) $\begin{pmatrix} -3 & 2 & 0 \\ 1 & -2 & -1 \\ -1 & 0 & 2 \\ 1 & 1 & -3 \end{pmatrix}$

3. Prove that if C_1 and C_2 are convex sets in $\mathbb{R}^n$, then the following sets are also convex.

   (a) $C_1 \cap C_2$

   (b) $C_1 + C_2 = \{\mathbf{x}_1 + \mathbf{x}_2 : \mathbf{x}_1 \in C_1, \mathbf{x}_2 \in C_2\}$

4. Let A be an $m \times n$ matrix. Prove that the set of maximin strategies for the row player of A is convex.

5. Let C be a convex set in $\mathbb{R}^n$ and $\mathbf{x}, \mathbf{y} \in C$. Let $\mathbf{z} \in \mathbb{R}^n$ be a point on the straight line joining $\mathbf{x}$ and $\mathbf{y}$ such that $\mathbf{z}$ is orthogonal to $\mathbf{x} - \mathbf{y}$.

   (a) Find $\mathbf{z}$ in terms of $\mathbf{x}$ and $\mathbf{y}$.

   (b) Suppose $\langle \mathbf{x}, \mathbf{y} \rangle < 0$. Prove that $\mathbf{z} \in C$.

6. Let A be an $m \times n$ matrix with column vectors $\mathbf{a}_1^T, \mathbf{a}_2^T, \cdots, \mathbf{a}_n^T$. Let $\nu_c(A)$ be the column value of A and let

   $$C = \text{Conv}(\{\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n, \mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_m\})$$

   where $\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_m\}$ is the standard basis for $\mathbb{R}^m$. Prove that if $\nu_c(A) \leq 0$, then $\mathbf{0} \in C$.

3 Bimatrix games

In this chapter, we study bimatrix game. A bimatrix game is a two-person game with perfect information. In a bimatrix game, two players, player I and player II, choose their strategies simultaneously. Then the payoffs to the players depend on the strategies used by the players. Unlike zero sum game, we have no assumption on the sum of payoffs to the players. We will first study non-cooperative games where the solutions are the Nash equilibria. Then we will study Nash's bargaining model and threat solution in cooperative game with nontransferable and transferable utilities respectively.

3.1 Nash equilibrium

A bimatrix game can be represented by two matrices, hence its name.

Definition 3.1.1 (Bimatrix game)**.** *The normal form of a* **bimatrix game** *is given by a pair of $m \times n$ matrices (A, B). The matrices A and B are payoff matrices for the row player (player I) and the column player (player II) respectively. Suppose the row player uses strategy $\mathbf{x} \in \mathcal{P}^m$ and the column player uses strategy $\mathbf{y} \in \mathcal{P}^n$. Then the payoff to the row player and column player are given by the payoff functions*

$$\begin{aligned} \pi(\mathbf{x}, \mathbf{y}) &= \mathbf{x}A\mathbf{y}^T \\ \rho(\mathbf{x}, \mathbf{y}) &= \mathbf{x}B\mathbf{y}^T \end{aligned}$$

respectively.

Definition 3.1.2. *The* **safety level***, or* **security level***, of the row player is*

$$\mu = \max_{\mathbf{x} \in \mathcal{P}^m} \min_{\mathbf{y} \in \mathcal{P}^n} \mathbf{x}A\mathbf{y}^T = \nu(A)$$

where $\nu(A)$ denotes the value of the matrix A when A is considered as the game matrix of a two-person zero sum game. The safety level of the column player is

$$\nu = \max_{\mathbf{y} \in \mathcal{P}^n} \min_{\mathbf{x} \in \mathcal{P}^m} \mathbf{x}B\mathbf{y}^T = \nu(B^T)$$

where $\nu(B^T)$ is the value of the transpose B^T of B.

Note that the value of a matrix is defined to be the maximum payoff that the row payoff may guarantee himself. The safety level of the column player of the bimatrix game (A, B) is the value ν_{B^T} of the transpose B^T of B, not the value of B.

Definition 3.1.3 (Nash equilibrium)**.** *Let* (A, B) *be a game bimatrix. We say that a pair of strategies* $(\mathbf{p}, \mathbf{q})$ *is an* **equilibrium pair***, or* **mixed Nash equilibrium***, or just* **Nash equilibrium***, for* (A, B) *if*

$$\mathbf{x}A\mathbf{q}^T \le \mathbf{p}A\mathbf{q}^T \text{ for any } \mathbf{x} \in \mathcal{P}^m$$

and

$$\mathbf{p}B\mathbf{y}^T \le \mathbf{p}B\mathbf{q}^T \text{ for any } \mathbf{y} \in \mathcal{P}^n$$

Example 3.1.4 (Prisoner dilemma)**.** *Let*

$$(A, B) = \begin{pmatrix} (-5,-5) & (-1,-10) \\ (-10,-1) & (-2,-2) \end{pmatrix}$$

which represents a version of the famous **prisoner dilemma***. The strategy pair* $(\mathbf{p}, \mathbf{q}) = ((1,0),(1,0))$ *is a Nash equilibrium. The Nash equilibrium is unique in this example.* □

Example 3.1.5 (Dating game)**.** *Consider*

$$(A, B) = \begin{pmatrix} (4,2) & (0,0) \\ (0,0) & (1,3) \end{pmatrix}.$$

It is an example of a **dating game**. There are two obvious Nash equilibria, which are pure Nash equilibria, namely $(\mathbf{p}, \mathbf{q}) = ((1,0),(1,0))$ and $((0,1),(0,1))$. The game has one more mixed Nash equilibrium (non-pure Nash equilibrium which is harder to find out. To see what it is, suppose the row player uses strategy $\mathbf{x} = (x, 1-x)$, where $0 \le x \le 1$. Then

$$\mathbf{x}B = (x, 1-x)\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} = (2x, 3-3x)$$

It means that the payoff to the column player is $2x$, and $3-3x$ if the column player constantly uses his 1st, and 2nd strategies respectively. Setting $2x = 3-3x$, we have $x = 0.6$ and

$$(0.6, 0.4)B = (0.6, 0.4)\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} = (1.2, 1.2)$$

Thus if the row player uses mixed strategy $(0.6, 0.4)$, then the payoff to the column player is always 1.2 no matter how the column player plays. Similarly suppose the column player uses $\mathbf{y} = (y, 1-y)$, $0 \leq y \leq 1$. Then

$$A\mathbf{y}^T = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y \\ 1-y \end{pmatrix} = \begin{pmatrix} 4y \\ 1-y \end{pmatrix}$$

It means that the payoff to the row player is $4y$, and $1-y$ if the row player constantly uses his 1st, and 2nd strategies respectively. Setting $4y = 1-y$, we have $y = 0.2$. Then

$$\begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0.2 \\ 0.8 \end{pmatrix} = \begin{pmatrix} 0.8 \\ 0.8 \end{pmatrix}$$

Thus if the column player uses mixed strategy $(0.2, 0.8)$, then the payoff to the row player is always 0.8 no matter how the row player plays. Therefore the strategy pair $(\mathbf{p}, \mathbf{q}) = ((0.6, 0.4), (0.2, 0.8))$ is a Nash equilibrium. In conclusion, the dating game has three Nash equilibria and we list them in the following table.

Nash equilibrium and the corresponding payoff pair

Row player's strategy $\mathbf{p}$	Column player's strategy $\mathbf{q}$	Payoff pair (π, ρ)
$(1, 0)$	$(1, 0)$	$(4, 2)$
$(0, 1)$	$(0, 1)$	$(1, 3)$
$(0.6, 0.4)$	$(0.2, 0.8)$	$(0.8, 1.2)$

$\square$

Note that in the third Nash equilibrium of the above example, the strategy for the row player $\mathbf{p} = (0.6, 0.4)$ is the minimax strategy for the column player of B^T, not the maximin strategy for the row player of A. That means what the row player should do is to fix the payoff to its opponent (the column player) to be 1.2 instead of guaranteeing the payoff to himself to be 0.8. Similarly, the strategy for the column player $\mathbf{q} = (0.2, 0.8)$ in this Nash equilibrium is the minimax strategy for the column player of A. So the column player should use a strategy to fix the row player's payoff instead of guaranteeing his own payoff.

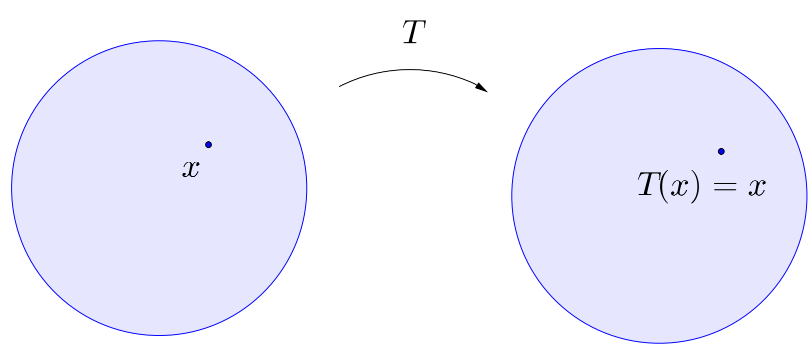
3.2 Nash's theorem

One of the most fundamental works in game theory is the following theorem of Nash which greatly extended the minimax theorem (Theorem 1.1.10). The theorem says that Nash equilibrium always exists in a non-cooperative game with finitely many players.

Theorem 3.2.1 (Nash's theorem)**.** *Every finite*[3] *game with finite number of players has at least one Nash equilibrium.*

Nash invoked the following celebrated theorem in topology to prove his theorem.

Theorem 3.2.2 (Brouwer's fixed-point theorem)**.** *Let X be a topological space which is homeomorphic to the closed unit ball $D^n = \{\mathbf{x} \in \mathbf{R}^n : \|\mathbf{x}\| \leq 1\}$. Then any continuous map $T : X \to X$ has at least one fixed-point, that is, there exists $x \in X$ such that $T(x) = x$.*

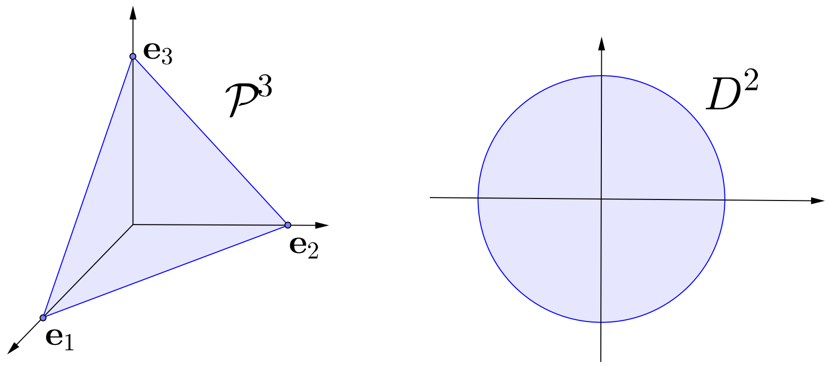


Remarks:

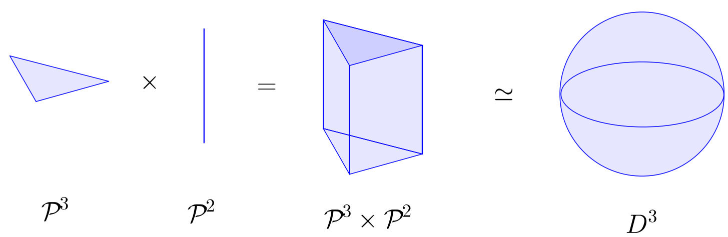
1. Two topological space X and Y are homeomorphic if there exists bijective map $\varphi : X \to Y$ such that both φ and its inverse φ^{-1} are continuous.

2. The set $\mathcal{P}^n = \{(x_1, x_2, \cdots, x_n) \in \mathbb{R}^n : x_1, \cdots, x_n \geq 0 \text{ and } x_1 + x_2 + \cdots + x_n = 1\}$ of probability vectors in $\mathbb{R}^n$ is homeomorphic to D^{n-1}.

[3] A game is finite if the number of strategies of each player is finite.



Moreover $\mathcal{P}^m \times \mathcal{P}^n$ is homeomorphic to D^{m+n-2}.



The proof of the Brouwer's fixed-point theorem is out of the propose and scope of this notes. Now we give the proof of Nash's theorem assuming the Brouwer's fixed-point theorem.

Proof of Nash's theorem. For simplicity, we consider two-person game only. The proof for the general case is similar. Let (A, B) be the game bimatrix of a two-person game. Define $T : \mathcal{P}^m \times \mathcal{P}^n \to \mathcal{P}^m \times \mathcal{P}^n$ by

$$T(\mathbf{x}, \mathbf{y}) = (\mathbf{u}, \mathbf{v}) = ((u_1, u_2, \cdots, u_m), (v_1, v_2, \cdots, v_n))$$

where for $k = 1, 2, \cdots, m$ and $l = 1, 2, \cdots, n$,

$$u_k = \frac{x_k + c_k}{1 + \sum\limits_{i=1}^{m} c_i} \text{ and } v_l = \frac{y_l + d_l}{1 + \sum\limits_{j=1}^{n} d_j}$$

and

$$\begin{aligned} c_k &= \max\{\pi(\mathbf{e}_k, \mathbf{y}) - \pi(\mathbf{x}, \mathbf{y}) = \mathbf{e}_k A \mathbf{y}^T - \mathbf{x} A \mathbf{y}^T, 0\} \\ d_l &= \max\{\rho(\mathbf{x}, \mathbf{e}_l) - \rho(\mathbf{x}, \mathbf{y}) = \mathbf{x} B \mathbf{e}_l^T - \mathbf{x} B \mathbf{y}^T, 0\} \end{aligned}$$

Here $\mathbf{e}_k$, $\mathbf{e}_l$ are vectors in the standard bases in $\mathbb{R}^m$, $\mathbb{R}^n$ respectively. By definition, c_k is the increase of payoff of the first player if the first player changes his strategy from $\mathbf{x}$ to $\mathbf{e}_k$ while the strategy of the second player remains at $\mathbf{y}$. However, if there is no increase, then we set $c_k = 0$. The numbers d_k are similarly defined. Note that $\mathbf{u} \in \mathcal{P}^m$ and $\mathbf{v} \in \mathcal{P}^n$ because

$$c_k, d_l \geq 0$$

and

$$\begin{aligned} \sum_{k=1}^{m} \left(\frac{x_k + c_k}{1 + \sum\limits_{i=1}^{m} c_i} \right) &= \frac{\sum\limits_{k=1}^{m} x_k + \sum\limits_{k=1}^{m} c_k}{1 + \sum\limits_{i=1}^{m} c_i} = \frac{1 + \sum\limits_{k=1}^{m} c_k}{1 + \sum\limits_{i=1}^{m} c_i} = 1 \\ \sum_{l=1}^{n} \left(\frac{y_l + d_l}{1 + \sum\limits_{j=1}^{n} d_j} \right) &= \frac{\sum\limits_{l=1}^{n} y_l + \sum\limits_{l=1}^{n} d_l}{1 + \sum\limits_{j=1}^{n} d_j} = \frac{1 + \sum\limits_{l=1}^{n} d_l}{1 + \sum\limits_{j=1}^{n} d_j} = 1 \end{aligned}$$

Now T is a continuous map from $\mathcal{P}^m \times \mathcal{P}^n$ to $\mathcal{P}^m \times \mathcal{P}^n$. By Brouwer's fixed-point theorem (Theorem 3.2.2), there exists $(\mathbf{p}, \mathbf{q}) \in \mathcal{P}^m \times \mathcal{P}^n$ such that

$$T(\mathbf{p}, \mathbf{q}) = (\mathbf{p}, \mathbf{q})$$

The proof of Nash's theorem is complete if we can prove that $(\mathbf{p}, \mathbf{q})$ is a Nash equilibrium. Suppose on the contrary that $(\mathbf{p}, \mathbf{q})$ is not a Nash equilibrium. Then either there exists $\mathbf{r} \in \mathcal{P}^m$ such that $\mathbf{r} A \mathbf{q}^T > \mathbf{p} A \mathbf{q}^T$ or there exists $\mathbf{s} \in \mathcal{P}^n$ such that $\mathbf{p} B \mathbf{s}^T > \mathbf{p} B \mathbf{q}^T$. Without loss of generality, we consider the former case. Write $\mathbf{r} = (r_1, r_2, \cdots, r_m)$. Since

$$\mathbf{p} A \mathbf{q}^T < \mathbf{r} A \mathbf{q}^T = \sum_{k=1}^{m} r_k \mathbf{e}_k A \mathbf{q}^T$$

and $\mathbf{r}$ is a probability vector, we see that there exists $1 \leq k \leq m$ such that

$$\mathbf{p}A\mathbf{q}^T < \mathbf{e}_k A\mathbf{q}^T$$

It follows that

$$c_k = \max\{\mathbf{e}_k A\mathbf{q}^T - \mathbf{p}A\mathbf{q}^T, 0\} > 0$$

and thus $\sum_{i=1}^{m} c_i > 0$. On the other hand, since

$$\mathbf{p}A\mathbf{q}^T = \sum_{i=1}^{m} p_i \mathbf{e}_i A\mathbf{q}^T$$

and $\mathbf{p}$ is a probability vector, there exists $1 \leq r \leq m$ such that $p_r > 0$ and

$$\mathbf{e}_r A\mathbf{q}^T \leq \mathbf{p}A\mathbf{q}^T$$

which implies, by the definition of c_r, that $c_r = 0$. Hence we have

$$\frac{p_r + c_r}{1 + \sum_{i=1}^{m} c_i} = \frac{p_r}{1 + \sum_{i=1}^{m} c_i} \leq \frac{p_r}{1 + c_k} < p_r$$

which contradicts that $(\mathbf{p}, \mathbf{q})$ is a fixed-point of T. Therefore $(\mathbf{p}, \mathbf{q})$ is a Nash equilibrium and the proof of Nash's theorem is complete. $\square$

We have seen in the proof of Nash's theorem that $(\mathbf{p}, \mathbf{q})$ is a Nash equilibrium if it is a fixed-point of T. As a matter of fact, the converse of this statement is also true. For if $(\mathbf{p}, \mathbf{q})$ is a Nash equilibrium, then $\mathbf{e}_i A\mathbf{q}^T \leq \mathbf{p}A\mathbf{q}^T$ for any $1 \leq i \leq m$. Thus $c_i = 0$ for any $1 \leq i \leq m$. Similarly $d_j = 0$ for any $1 \leq j \leq n$. Therefore $T(\mathbf{p}, \mathbf{q}) = (\mathbf{p}, \mathbf{q})$.

To find Nash equilibria of a 2×2 game bimatrix (A, B), we may let $\mathbf{x} = (x, 1-x)$, $\mathbf{y} = (y, 1-y)$ and consider the payoff functions

$$\begin{aligned} \pi(x, y) &= \pi(\mathbf{x}, \mathbf{y}) = \mathbf{x}A\mathbf{y}^T \\ \rho(x, y) &= \rho(\mathbf{x}, \mathbf{y}) = \mathbf{x}B\mathbf{y}^T \end{aligned}$$

Define

$$\begin{aligned} P &= \{(x, y) : \pi(x, y) \text{ attains its maximum at } x \text{ for fixed } y.\} \\ Q &= \{(x, y) : \rho(x, y) \text{ attains its maximum at } y \text{ for fixed } x.\} \end{aligned}$$

Then $(\mathbf{x}, \mathbf{y})$ is a Nash equilibrium if and only if $(x, y) \in P \cap Q$.

Example 3.2.3 (Prisoner dilemma)**.** *Consider the prisoner dilemma (Example 3.1.4) with bimatrix*

$$(A, B) = \begin{pmatrix} (-5,-5) & (-1,-10) \\ (-10,-1) & (-2,-2) \end{pmatrix}$$

The payoff to the row player is given by

$$\begin{aligned} \pi(x, y) &= (x, 1-x) \begin{pmatrix} -5 & -1 \\ -10 & -2 \end{pmatrix} \begin{pmatrix} y \\ 1-y \end{pmatrix} \\ &= (x, 1-x) \begin{pmatrix} -4y-1 \\ -8y-2 \end{pmatrix} \end{aligned}$$

Since $-8y - 2 < -4y - 1$ for any $0 \le y \le 1$, we have

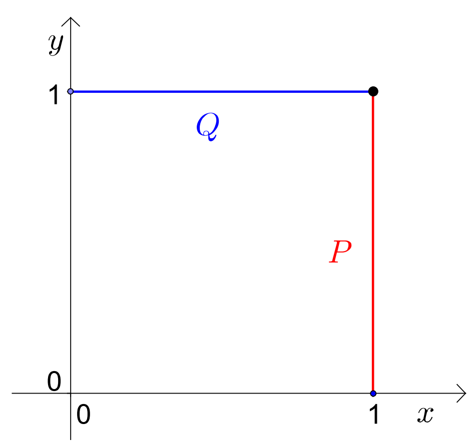
$$P = \{(1, y) : 0 \le y \le 1\}$$

On the other hand,

$$\begin{aligned} \rho(x, y) &= (x, 1-x) \begin{pmatrix} -5 & -10 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} y \\ 1-y \end{pmatrix} \\ &= (-4x-1, -8x-2) \begin{pmatrix} y \\ 1-y \end{pmatrix} \end{aligned}$$

Since $-8x - 2 < -4x - 1$ for any $0 \le x \le 1$, we have

$$Q = \{(x, 1) : 0 \le p \le 1\}$$



Now

$$P \cap Q = \{(1,1)\}$$

Therefore the game has a unique Nash equilibrium $(\mathbf{p}, \mathbf{q}) = ((1,0),(1,0))$.$\square$

Example 3.2.4 (Dating game)**.** *Consider the dating game (Example 3.1.5) with bimatrix*

$$(A,B) = \begin{pmatrix} (4,2) & (0,0) \\ (0,0) & (1,3) \end{pmatrix}$$

We have

$$\begin{aligned} \pi(x,y) &= (x, 1-x)\begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y \\ 1-y \end{pmatrix} \\ &= (x, 1-x)\begin{pmatrix} 4y \\ 1-y \end{pmatrix} \end{aligned}$$

Now

$$\begin{cases} 4y < 1-y & \text{if} \quad 0 \le y < \dfrac{1}{5} \\ 4y = 1-y & \text{if} \quad y = \dfrac{1}{5} \\ 4y > 1-y & \text{if} \quad \dfrac{1}{5} < y \le 1 \end{cases}$$

Thus

$$P = \left\{(x,y) : \left(x = 0 \wedge 0 \le y < \frac{1}{5}\right) \vee \left(0 \le x \le 1 \wedge y = \frac{1}{5}\right) \vee \left(x = 1 \wedge \frac{1}{5} < y \le 1\right)\right\}$$

On the other hand,

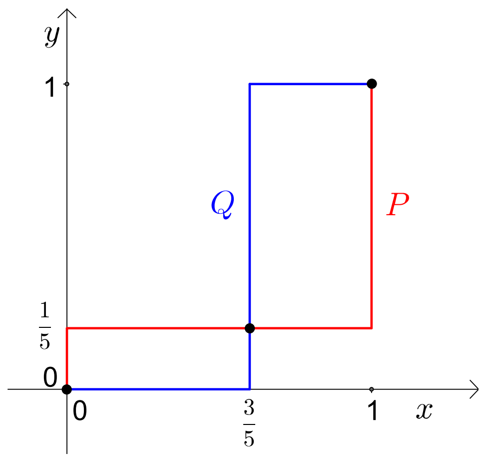
$$\begin{aligned} \rho(x,y) &= (x, 1-x)\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} y \\ 1-y \end{pmatrix} \\ &= (2x, 3-3x)\begin{pmatrix} y \\ 1-y \end{pmatrix} \end{aligned}$$

Now

$$\begin{cases} 2x < 3-3x & \text{if} \quad 0 \le x < \dfrac{3}{5} \\ 2x = 3-3x & \text{if} \quad x = \dfrac{3}{5} \\ 2x > 3-3x & \text{if} \quad \dfrac{3}{5} < x \le 1 \end{cases}$$

Thus

$$Q = \left\{(x,y) : \left(0 \le x < \frac{3}{5} \wedge y = 0\right) \vee \left(x = \frac{3}{5} \wedge 0 \le y \le 1\right) \vee \left(\frac{3}{5} < x \le 1 \wedge y = 1\right)\right\}$$



Now

$$P \cap Q = \left\{(0,0), (1,1), \left(\frac{3}{5}, \frac{1}{5}\right)\right\}$$

Therefore the game has three Nash equilibria

$$(\mathbf{p}, \mathbf{q}) = ((1,0),(1,0)), ((0,1),(0,1)), \left(\frac{3}{5}, \frac{2}{5}\right), \left(\left(\frac{1}{5}, \frac{4}{5}\right)\right).$$

We list the associated payoff pairs in the following table.

$\mathbf{p}$	$\mathbf{q}$	(π, ρ)
$(1,0)$	$(1,0)$	$(4,2)$
$(0,1)$	$(0,1)$	$(1,3)$
$\left(\frac{3}{5}, \frac{2}{5}\right)$	$\left(\frac{1}{5}, \frac{4}{5}\right)$	$\left(\frac{4}{5}, \frac{6}{5}\right)$

$\square$

Definition 3.2.5. *Let (A, B) be a game bimatrix.*

1. *We say that two Nash equilibria* $(\mathbf{p},\mathbf{q})$ *and* $(\mathbf{p}',\mathbf{q}')$ *are* **interchangeable** *if* $(\mathbf{p}',\mathbf{q})$ *and* $(\mathbf{p},\mathbf{q}')$ *are also Nash equilibria.*

2. *We say that two Nash equilibria* $(\mathbf{p},\mathbf{q})$ *and* $(\mathbf{p}',\mathbf{q}')$ *are equivalent if*

$$\pi((\mathbf{p},\mathbf{q}),\rho(\mathbf{p},\mathbf{q})) = \pi((\mathbf{p}',\mathbf{q}'),\rho(\mathbf{p}',\mathbf{q}'))$$

3. *We say that a bimatrix game* (A,B) *is* **solvable in the Nash sense** *if any two Nash equilibria are interchangeable and equivalent.*

For the prisoner dilemma (Example 3.2.3), there is only one Nash equilibrium. Thus the prisoner dilemma is solvable in the Nash sense. For the dating game (Example 3.2.4), there are three Nash equilibria which are not interchangeable. So the dating game is not solvable in the Nash sense.

Example 3.2.6. *Solve the game bimatrix*

$$(A,B) = \begin{pmatrix} (1,4) & (5,1) \\ (4,2) & (3,3) \end{pmatrix}$$

Solution. Consider

$$A\mathbf{y}^T = \begin{pmatrix} 1 & 5 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} y \\ 1-y \end{pmatrix} = \begin{pmatrix} -4y+5 \\ y+3 \end{pmatrix}$$

Now

$$\begin{cases} -4y+5 > y+3 & \text{if} \quad 0 \le y < \dfrac{2}{5} \\ -4y+5 = y+3 & \text{if} \quad y = \dfrac{2}{5} \\ -4y+5 < y+3 & \text{if} \quad \dfrac{2}{5} < y \le 1 \end{cases}$$

We see that

$$P = \left\{(x,y) : \left(x = 0 \wedge \frac{2}{5} < y \le 1\right) \vee \left(0 \le x \le 1 \wedge y = \frac{2}{5}\right) \vee \left(x = 1 \wedge 0 \le y < \frac{2}{5}\right)\right\}$$

On the other hand

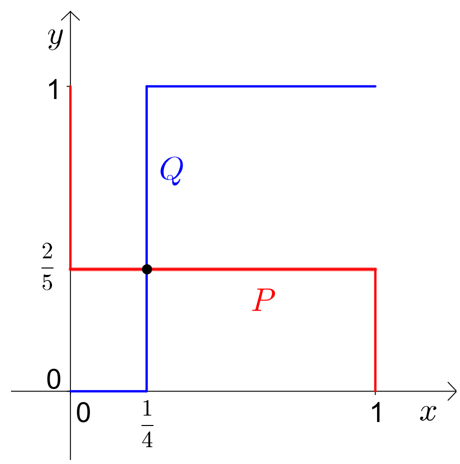
$$\mathbf{x}B = (x, 1-x)\begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix} = (2x+2, -2x+3)$$

and

$$\begin{cases} 2x+2 < -2x+3 & \text{if} \quad 0 \le x < \dfrac{1}{4} \\ 2x+2 = -2x+3 & \text{if} \quad x = \dfrac{1}{4} \\ 2x+2 > -2x+3 & \text{if} \quad \dfrac{1}{4} < x \le 1 \end{cases}$$

We see that

$$Q = \left\{(x,y) : \left(0 \le x < \frac{1}{4} \wedge y = 0\right) \vee \left(x = \frac{1}{4} \wedge 0 \le y \le 1\right) \vee \left(\frac{1}{4} < x \le 1 \wedge y = 1\right)\right\}$$



Now

$$P \cap Q = \left\{\left(\frac{1}{4}, \frac{2}{5}\right)\right\}$$

Therefore the game has Nash equilibrium

$$(\mathbf{p}, \mathbf{q}) = \left(\left(\frac{1}{4}, \frac{3}{4}\right), \left(\frac{2}{5}, \frac{3}{5}\right)\right)$$

and is solvable in the Nash sense since the Nash equilibrium is unique. $\square$

3.3 Nash bargaining model

A bimatrix game can be played as a cooperative game with **non-transferable utility**. This means the players may make agreements on what strategies they are going to use. However they are not allowed to share the payoffs they obtained in the game. In such a game, players may use joint strategies.

Definition 3.3.1. *Let (A, B) be an $m \times n$ bimatrix of a two-person game.*

1. *A* **joint strategy** *of (A, B) is an $m \times n$ matrix*

$$P = \begin{pmatrix} p_{11} & \cdots & p_{1n} \\ \vdots & \ddots & \vdots \\ p_{m1} & \cdots & p_{mn} \end{pmatrix}$$

which satisfies

(i) $p_{ij} \geq 0$ *for any* $i = 1, 2, \cdots, m$ *and* $j = 1, 2, \cdots, n$

(ii) $\sum_{i=1}^{m} \sum_{j=1}^{n} p_{ij} = 1$

In other words, P is a joint strategy if it is a **probability matrix**. *The set of all $m \times n$ probability matrices is denoted by*

$$\mathcal{P}^{m \times n} = \{P = [p_{ij}] : p_{ij} \geq 0 \text{ and } \sum p_{ij} = 1\}$$

In particular, if $\mathbf{p} = (p_1, \cdots, p_m) \in \mathcal{P}^m$ and $\mathbf{q} = (q_1, \cdots, q_n) \in \mathcal{P}^n$, then

$$\mathbf{p}^T \mathbf{q} = \begin{pmatrix} p_1 q_1 & \cdots & p_1 q_n \\ \vdots & \ddots & \vdots \\ p_m q_1 & \cdots & p_m q_n \end{pmatrix} \in \mathcal{P}^{m \times n}$$

is a joint strategy. In this case, the row player uses strategy $\mathbf{p}$ and the column player uses strategy $\mathbf{q}$ independently. Not all joint strategies are of this form. For example

$$\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

cannot be expressed as the form $\mathbf{p}^T \mathbf{q}$. When this joint strategy is used, the players may flip a coin and both use their first strategies if a 'head' is obtained and both use their second strategies if a 'tail' is obtained.

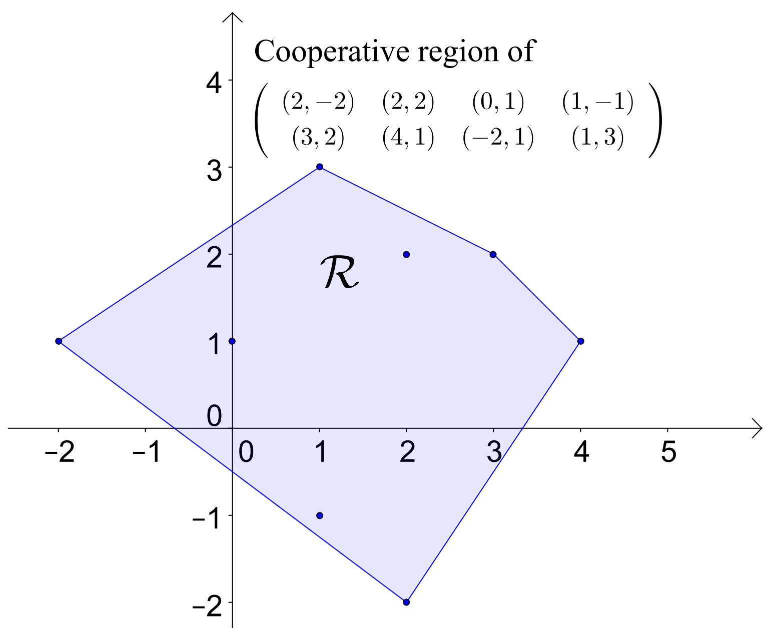
2. *For joint strategy $P = [p_{ij}] \in \mathcal{P}^{m\times n}$, the payoff u to the row player and the payoff v to the column player are given by the payoff pair*

$$\begin{aligned} (u(P), v(P)) &= \left(\sum_{i,j} a_{ij}p_{ij}, \sum_{i,j} b_{ij}p_{ij}\right) \\ &= \sum_{i,j} p_{ij}(a_{ij}, b_{ij}) \end{aligned}$$

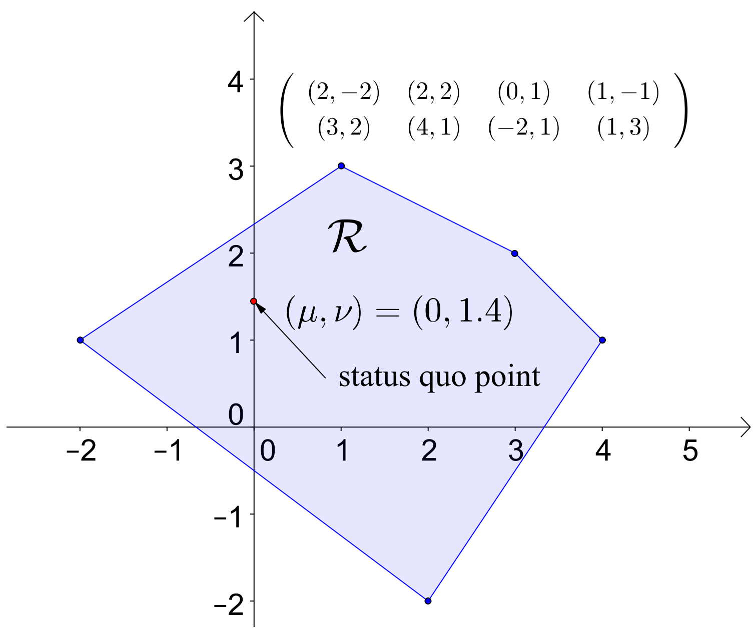
3. *The* **cooperative region** *of (A, B) is the set of all feasible payoff pairs*

$$\begin{aligned} \mathcal{R} &= \{(u(P), v(P)) \in \mathbb{R}^2 : P \in \mathcal{P}^{m\times n}\} \\ &= \left\{(u, v) \in \mathbb{R}^2 : (u, v) = \sum_{i,j} p_{ij}(a_{ij}, b_{ij}) \text{ for some } [p_{ij}] \in \mathcal{P}^{m\times n}\right\} \end{aligned}$$

In other words, the cooperative region $\mathcal{R}$ is the convex hull of the set of points $\{(a_{ij}, b_{ij}) : 1 \le i \le m, 1 \le j \le n\}$ in $\mathbb{R}^2$. Note that $\mathcal{R}$ is a closed convex polygon in $\mathbb{R}^2$.



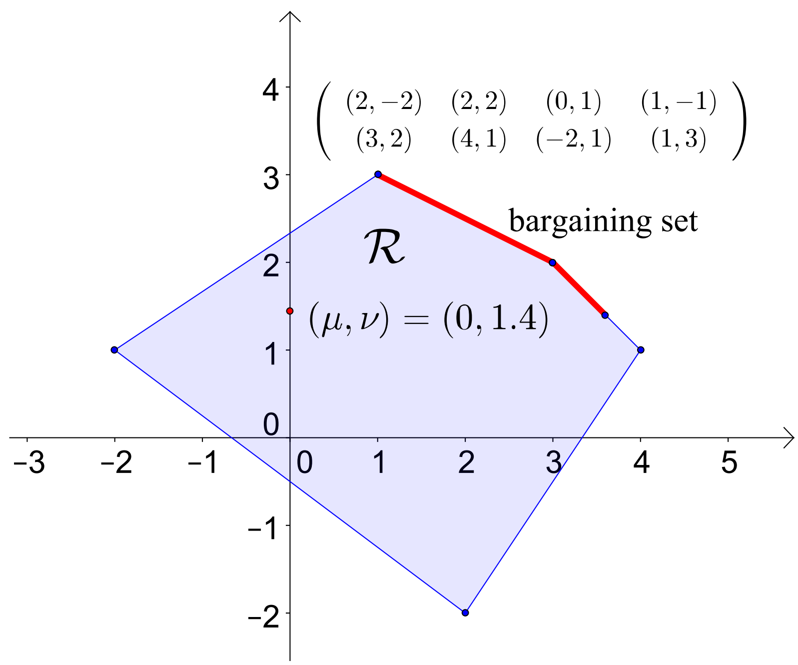
4. *The* **status quo point** *is the payoff pair* (μ, ν) *for the players associated to the solution of the game when* (A, B) *is considered as a non-cooperative game. In other words, the status quo point is the payoffs that the players may expect if the negotiations break down. Unless otherwise specified, we will take* $(\mu, \nu) = (\nu(A), \nu(B^T))$ *to be the status quo point where* $v(A)$ *and* $\nu(B^T)$ *are the values of* A *and the transpose* B^T *of* B *respectively.*



5. *We say that a payoff pair* (u, v) *is* **Pareto optimal** *if* $u' \geq u$, $v' \geq v$ *and* $(u', v') \in \mathcal{R}$ *implies* $(u', v') = (u, v)$ *where* $\mathcal{R}$ *is the cooperative region.*

6. *The* **bargaining set** *of* (A, B) *is the set of Pareto optimal payoff pairs* $(u, v) \in \mathcal{R}$ *such that* $u \geq \mu$ *and* $v \geq \nu$ *where* (μ, ν) *is the status quo point. In other words, the bargaining set is*

$$\{(u, v) \in \mathcal{R} : u \geq \mu, v \geq \nu \text{ and } (u, v) \text{ is Pareto optimal}\}$$



When the status quo point is not Pareto optimal, the two players of the game would have a tendency to cooperate. The **bargaining problem** is a problem to understand how the players should cooperate in this situation. Nash proposed that the solution to the bargaining problem is a function, called the arbitration function, depending only on the cooperative region $\mathcal{R}$ and the status quo point $(\mu,\nu) \in \mathcal{R}$, which satisfies certain properties called Nash bargaining axioms.

Definition 3.3.2 (Nash bargaining axioms)**.** *An* **arbitration function** *is a function* $(\alpha,\beta) = A(\mathcal{R},(\mu,\nu))$ *defined for a closed and bounded convex set* $\mathcal{R} \subset \mathbb{R}^2$ *(cooperative region) and a point* $(\mu,\nu) \in \mathcal{R}$ *(status quo point) such that the following* **Nash bargaining axioms** *are satisfied.*

1. *(Individual rationality)* $\alpha \geq \mu$ *and* $\beta \geq \nu$*.*

2. *(Pareto optimality) For any* $(u,v) \in \mathcal{R}$*, if* $u \geq \alpha$ *and* $v \geq \nu$*, then* $(u,v) = (\alpha,\beta)$*.*

3. *(Feasibility)* $(\alpha,\beta) \in \mathcal{R}$*.*

4. *(Independence of irrelevant alternatives) If $\mathcal{R}' \subset \mathcal{R}$, $(\mu, \nu) \in \mathcal{R}'$ and $(\alpha, \beta) = A(\mathcal{R}, (\mu, \nu)) \in \mathcal{R}'$, then $A(\mathcal{R}', (\mu, \nu)) = (\alpha, \beta) = A(\mathcal{R}, (\mu, \nu))$.*

5. *(Invariant under linear transformation) Let $a, b, c, d \in \mathbb{R}$ be any real numbers with $a, c > 0$. Let $\mathcal{R}' = \{(au + b, cv + d) : (u, v) \in \mathcal{R}\}$ and $(\mu', \nu') = (a\mu + b, c\nu + d)$. Then $A(\mathcal{R}', (\mu', \nu')) = (a\alpha + b, c\beta + d)$.*

6. *(Symmetry) Suppose $\mathcal{R}$ is symmetry, that is $(u, v) \in \mathcal{R}$ implies $(v, u) \in \mathcal{R}$, and $\mu = \nu$. Then $\alpha = \beta$.*

Theorem 3.3.3 (Nash bargaining solution)**.** *There exists a unique arbitration function $A(\mathcal{R}, (\mu, \nu))$ for closed and bounded convex set $\mathcal{R}$ and $(\mu, \nu) \in \mathcal{R}$ which satisfies the Nash bargaining axioms.*

Before proving Theorem 3.3.3, first we prove a lemma.

Lemma 3.3.4. *Let $\mathcal{R} \subset \mathbb{R}^2$ be any closed and bounded convex set and $(\mu, \nu) \in \mathcal{R}$. Let*

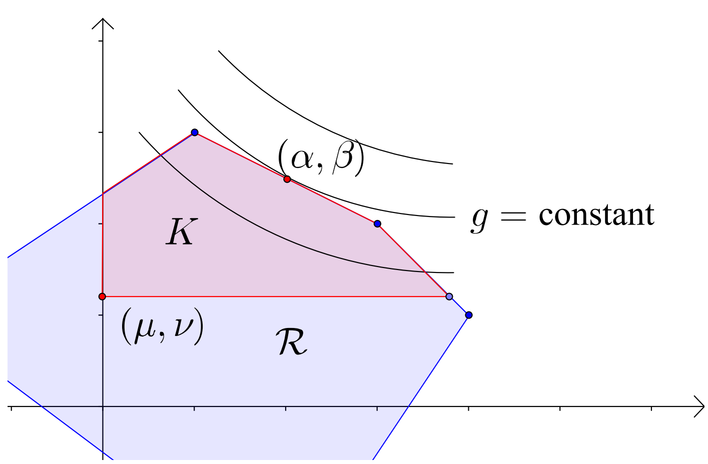
$$K = \{(u, v) \in \mathcal{R} : u \geq \mu, v \geq \nu\}$$

Let $g : K \to \mathbb{R}$ be the function defined by

$$g(u, v) = (u - \mu)(v - \nu) \text{ for } (u, v) \in K$$

Suppose $U = \{(u, v) \in K : u > \mu, v > \nu\} \neq \emptyset$. Then there exists unique $(\alpha, \beta) \in K$ such that

$$g(\alpha, \beta) = \max_{(u,v) \in K} g(u, v)$$



Proof. Since g is continuous and K is closed and bounded, g attains its maximum at some point $(\alpha, \beta) \in K$ and let

$$M = \max_{(u,v)\in K} g(u, v)$$

be the maximum value of g on K. We are going to prove by contradiction that the maximum point of g on K is unique. Suppose on the contrary that there exists $(\alpha', \beta') \in K$ with $(\alpha', \beta') \neq (\alpha, \beta)$ such that

$$g(\alpha', \beta') = g(\alpha, \beta) = M$$

Then either $\alpha' > \alpha$ and $\beta' < \beta$, or $\alpha' < \alpha$ and $\beta' > \beta$. In both case we have $(\alpha - \alpha')(\beta' - \beta) > 0$. Observe that the mid-point $(\frac{\alpha+\alpha'}{2}, \frac{\beta+\beta'}{2})$ of (α, β) and (α', β') lies in K since K is convex. On the other hand, the value of g at $(\frac{\alpha+\alpha'}{2}, \frac{\beta+\beta'}{2})$ is

$$\begin{aligned}
& g\left(\frac{\alpha+\alpha'}{2}, \frac{\beta+\beta'}{2}\right) \\
= & \left(\frac{\alpha+\alpha'}{2} - \mu, \frac{\beta+\beta'}{2} - \nu\right) \\
= & \frac{1}{4}\left((\alpha-\mu) + (\alpha'-\mu)\right)\left((\beta-\nu) + (\beta'-\nu)\right) \\
= & \frac{1}{4}((\alpha-\mu)(\beta-\nu) + (\alpha-\mu)(\beta'-\nu) \\
& +(\alpha'-\mu)(\beta-\nu) + (\alpha'-\mu)(\beta'-\nu)) \\
= & \frac{1}{4}((\alpha-\mu)(\beta-\nu) + (\alpha-\mu)((\beta'-\beta) + (\beta-\nu)) \\
& +(\alpha'-\mu)((\beta-\beta') + (\beta'-\nu)) + (\alpha'-\mu)(\beta'-\nu)) \\
= & \frac{1}{4}(2(\alpha-\mu)(\beta-\nu) + (\alpha-\mu)(\beta'-\beta) \\
& +(\alpha'-\mu)(\beta-\beta') + 2(\alpha'-\mu)(\beta'-\nu)) \\
= & \frac{1}{4}\left(2g(\alpha,\beta) + (\alpha-\alpha')(\beta'-\beta) + 2g(\alpha',\beta')\right) \\
= & \frac{1}{4}\left(2M + (\alpha-\alpha')(\beta'-\beta) + 2M\right) \\
> & M
\end{aligned}$$

This contradicts that the maximum value of g on K is M. Therefore g attains its maximum at a unique point. □

Proof of existence of arbitration function. For any closed and bounded convex set $\mathcal{R}$ and $(\mu,\nu) \in \mathcal{R}$, let $K = \{(u,v) \in \mathcal{R} : u \geq \mu, v \geq \nu\}$, $U = \{(u,v) \in \mathcal{R} : u > \mu, v > \nu\}$ and define $(\alpha,\beta) = A(\mathcal{R},(\mu,\nu))$ as follows:

1. If $U \neq \emptyset$, then $(\alpha,\beta) = A(\mathcal{R},(\mu,\nu)) \in K$ is the unique maximum point of $g(u,v) = (u-\mu)(v-\nu)$ in K, that is
$$g(\alpha,\beta) = \max_{(u,v)\in K} g(u,v)$$
2. If $U = \emptyset$, then $(\alpha,\beta) = A(\mathcal{R},(\mu,\nu)) \in K$ is the unique maximum point of $u+v$ on K, that is
$$\alpha+\beta = \max_{(u,v)\in K}(u+v)$$

We are going to prove that the function $A(\mathcal{R},(\mu,\nu))$ satisfies the Nash bargaining axioms. We prove only for the first case $U \neq \emptyset$ and the second case is obvious.

1. (Individual rationality) It follows by the definition that $(\alpha,\beta) \in K$ and we have $\alpha \geq \mu$ and $\beta \geq \nu$.
2. (Pareto optimality) Suppose there exists $(\alpha',\beta') \in \mathcal{R}$ such that $\alpha' \geq \alpha$ and $\beta' \geq \beta$. Then $g(\alpha',\beta') \geq g(\alpha,\beta)$ which implies that $(\alpha',\beta') = (\alpha,\beta)$ since the maximum point of g on K is unique.
3. (Feasibility) Since $(\alpha,\beta) \in K \subset \mathcal{R}$ by definition, we have $(\alpha,\beta) \in \mathcal{R}$.
4. (Independence of irrelevant alternatives) Suppose $\mathcal{R}' \subset \mathcal{R}$ is a subset of $\mathcal{R}$ which contains both (μ,ν) and (α,β). Since g attains its maximum at (α,β) on K, it also attains its maximum at (α,β) on $K' = K \cap \mathcal{R}'$. Thus
$$A(\mathcal{R}',(\mu,\nu)) = (\alpha,\beta) = A(\mathcal{R},(\mu,\nu))$$
5. (Invariant under linear transformation) Let $a,b,c,d \in \mathbb{R}$ with $a,c > 0$. Let $\mathcal{R}' = \{(u',v') = (au+b, cv+d) : (u,v) \in \mathcal{R}\}$ and $(\mu',\nu') = (a\mu+b, c\nu+d)$. Then
$$\begin{aligned} g'(u',v') &= (u'-\mu')(v'-\nu') \\ &= ((au+b)-(a\mu+b))((cv+d)-(c\nu+d)) \\ &= ac(u-\mu)(v-\nu) \\ &= acg(u,v) \end{aligned}$$

Hence g' attains its maximum at $(\alpha', \beta') = (a\alpha + b, c\beta + d)$ on $K' = \{(u', v') = (au + b, cv + d) : (u, v) \in K\}$ since g attains its maximum at (α, β) on K. Therefore $A(\mathcal{R}', (\mu, \nu)) = (\alpha', \beta')$.

6. (Symmetry) Suppose $\mathcal{R}$ is symmetric and $\mu = \nu$. Then

$$g(u, v) = (u - \mu)(v - \mu) = g(v, u)$$

and $(v, u) \in K$ if and only if $(u, v) \in K$. Thus if g attains its maximum at (α, β) on K, then g also attains its maximum at (β, α) on K. By uniqueness of maximum point of g on K, we see that $(\beta, \alpha) = (\alpha, \beta)$ which implies $\alpha = \beta$.

□

Proof of uniqueness of arbitration function. Suppose $A'(\mathcal{R}, (\mu, \nu))$ is another arbitration function satisfying the Nash bargaining axioms. Let $\mathcal{R}$ be a closed and bounded convex set and $(\mu, \nu) \in \mathcal{R}$. By applying a linear transformation, we may assume that $(\mu, \nu) = (0, 0)$ and $(\alpha, \beta) = A(\mathcal{R}, (0, 0)) = (0, 0), (1, 0), (0, 1)$ or $(1, 1)$. We are going to prove that $A'(\mathcal{R}, (0, 0)) = (\alpha, \beta)$.

Case 1. $(\alpha, \beta) = (0, 0)$:

In this case $K = \{(0, 0)\}$ and we have $A'(\mathcal{R}, (0, 0))$ since $(\alpha, \beta) \in K$.

Case 2. $(\alpha, \beta) = (1, 0)$ or $(0, 1)$:

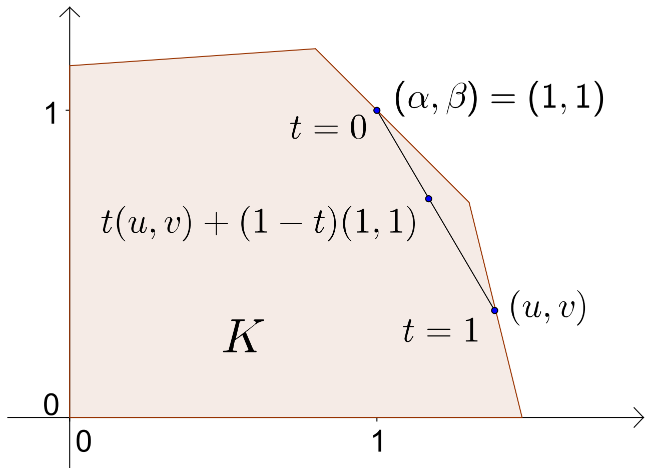
We consider the case for $(\alpha, \beta) = (1, 0)$ and the other case is similar. By definition of (α, β), we must have $K = \{(u, 0) : 0 \leq u \leq 1\}$. By the individual rationality, we have $A'(\mathcal{R}, (0, 0)) \in K$. By Pareto optimality, we have $A'(\mathcal{R}, (0, 0)) = (1, 0)$.

Case 3. $(\alpha, \beta) = (1, 1)$:

First we claim that $u + v \leq 2$ for any $(u, v) \in K$. We prove the claim by contradiction. Suppose there exists $(u, v) \in K$ such that $u + v > 2$. Then for any $0 \leq t \leq 1$, we have

$$t(u, v) + (1 - t)(1, 1) = ((u - 1)t + 1, (v - 1)t + 1) \in K$$

since K is convex. Let $g(t)$ be the value of g at the point $t(u, v) + (1 - t)(1, 1) \in K$ lying on the line segment joining $(1, 1)$ and (u, v).



Then

$$\begin{aligned} g(t) &= g(1 + (u-1)t, 1 + (v-1)t) \\ &= ((u-1)t + 1)((v-1)t + 1) \\ &= (u-1)(v-1)t^2 + (u+v-2)t + 1 \end{aligned}$$

We have

$$g'(t) = 2(u-1)(v-1)t + u + v - 2$$

which implies

$$g'(0) = u + v - 2 > 0$$

It follows that there exists $0 < t \leq 1$ such that

$$g(t) > g(0) = g(1, 1)$$

which contradicts that g attains its maximum at $(1, 1)$ on K. Hence we proved the claim that $u + v \leq 2$ for any $(u, v) \in K$. Now let $\mathcal{R}'$ be the convex hull of $\{(u, v) : (u, v) \in \mathcal{R} \text{ or } (v, u) \in \mathcal{R}\}$. Then $u' + v' \leq 2$ for any $(u', v') \in \mathcal{R}'$ since $u + v \leq 2$ for any $(u, v) \in \mathcal{R}$. By symmetry, we have $A'(\mathcal{R}', (0, 0)) = (\alpha', \alpha')$ for some $(\alpha', \alpha') \in \mathcal{R}'$. Now $\alpha' \leq 1$ since $\alpha' + \alpha' \leq 2$. Since $(1, 1) \in K \subset \mathcal{R}'$, we have $A'(\mathcal{R}', (0, 0)) = (1, 1)$ by Pareto optimality. Therefore $A'(\mathcal{R}, (0, 0)) = (1, 1)$ by independence of irrelevant alternative.

This completes the proof that $A'(\mathcal{R},(\mu,\nu)) = A(\mathcal{R},(\mu,\nu))$ for any closed and bounded convex set $\mathcal{R}$ and any point $(\mu,\nu) \in \mathcal{R}$. $\square$

Example 3.3.5 (Dating game). *Consider the dating game given by the bimatrix*

$$(A,B) = \begin{pmatrix} (4,2) & (0,0) \\ (0,0) & (1,3) \end{pmatrix}$$

We use $(\mu,\nu) = (\nu(A), \nu(B^T)) = (\frac{4}{5}, \frac{6}{5})$ as the status quo point (see Example 3.1.5). We need to find the payoff pair on

$$K = \left\{ (u,v) \in \mathcal{R} : u \geq \frac{4}{5}, v \geq \frac{6}{5} \right\}$$

so that the function

$$g(u,v) = \left(u - \frac{4}{5}\right)\left(v - \frac{4}{5}\right)$$

attains its maximum. Now any payoff pair (u,v) along the line segment joining $(1,3)$ and $(4,2)$ satisfies

$$\begin{aligned} v - 3 &= -\frac{1}{3}(u-1) \\ v &= -\frac{1}{3}u + \frac{10}{3} \end{aligned}$$

Thus

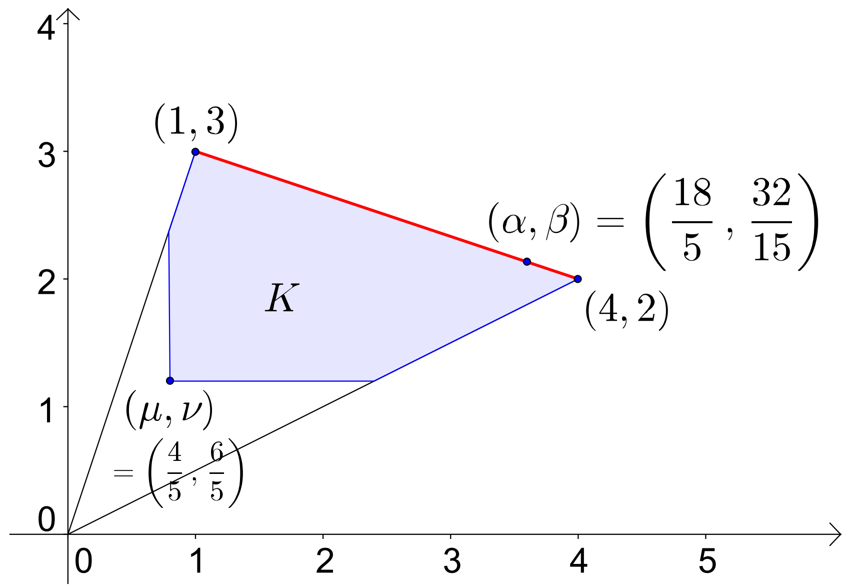
$$\begin{aligned} g(u,v) &= \left(u - \frac{4}{5}\right)\left(v - \frac{6}{5}\right) \\ &= \left(u - \frac{4}{5}\right)\left(-\frac{1}{3}u + \frac{32}{15}\right) \\ &= -\frac{1}{3}u^2 + \frac{12}{5}u - \frac{128}{75} \end{aligned}$$

attains its maximum when

$$u = \frac{18}{5} \text{ and } v = \frac{32}{15}$$

Since this payoff pair lies on the line segment joining $(1,3)$ and $(4,2)$, the arbitration pair of the game with status quo point $(\mu,\nu) = (\frac{4}{5}, \frac{6}{5})$ is

$$(\alpha,\beta) = \left(\frac{18}{5}, \frac{32}{15}\right)$$



$\square$

To find the arbitration pair, one may use the fact that if $g(u,v) = (u - \mu)(v - \nu)$ attains it maximum at the point (α, β) over the line joining (u_0, v_0) and (u_1, v_1), then the slope of the line joining (α, β) and (μ, ν) would be equal to the negative of the slope of line joining (u_0, v_0) and (u_1, v_1). Using this fact, one may see easily that (α, β) satisfies

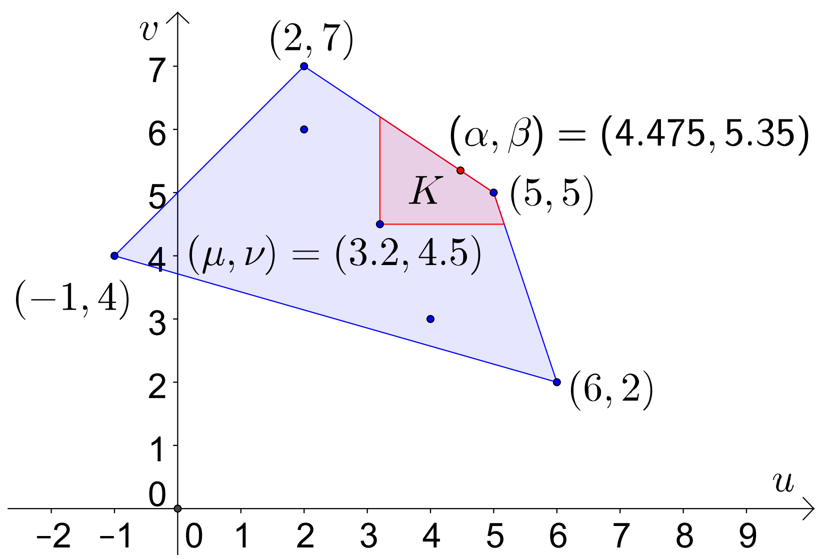
$$\begin{cases} \beta - v_0 = \dfrac{v_1 - v_0}{u_1 - u_0}(\alpha - u_0) \\ \beta - \nu = -\dfrac{v_1 - v_0}{u_1 - u_0}(\alpha - \mu) \end{cases}$$

Hence if the payoff pair (α, β) obtained by solving the above system of equations lies on the line segment joining (u_0, v_0) and (u_1, v_1), which implies that (α, β) lies on the bargaining set, then (α, β) is the arbitrary pair.

Example 3.3.6. *Let*

$$(A, B) = \begin{pmatrix} (2,6) & (6,2) & (-1,4) \\ (4,3) & (2,7) & (5,5) \end{pmatrix}$$

The reader may check that the values of A, B^T are 3.2, 4.5 respectively and we use $(\mu, \nu) = (3.2, 4.5)$ as the status quo point. We need to consider two line segments.



1. The line segment joining $(5,5)$ and $(6,2)$:

   The equation of the line segment is given by $v = -3u + 20$. The value of $g(u,v)$ along the line segment is

$$\begin{aligned} g(u,v) &= (u-3.2)(v-4.5) \\ &= (u-3.2)(-3u+15.5) \\ &= -3u^2 + 25.1u + 49.6 \end{aligned}$$

   which attain its maximum at $(\frac{251}{60}, \frac{149}{20})$. Since this payoff pair lies outside the line segment joining $(5,5)$ and $(6,2)$ and thus lies outside K, we know that the arbitration pair does not lie on the line segment joining $(5,5)$ and $(6,2)$.

2. The line segment joining $(2,7)$ and $(5,5)$:

   The slope of the line joining $(2,7)$ and $(5,5)$ is $-\frac{2}{3}$. To find the maximum point of $g(u,v)$ along the line joining $(2,7)$ and $(5,5)$, we may solve

$$\begin{cases} v - 7 = -\dfrac{2}{3}(u-2) \\ v - 4.5 = \dfrac{2}{3}(u-3.2) \end{cases}$$

which gives $(u, v) = (4.475, 5.35)$. Since this payoff pair lies on the line segment joining $(2, 7)$ and $(5, 5)$, we conclude that the arbitration pair is $(\alpha, \beta) = (4.475, 5.35)$.

□

3.4 Threat solution

In this section, we study two-person **cooperative games with transferable utility**. We assume that the players are 'rational' in the sense that, given a choice between two possible outcomes of differing personal utility, each player will select the one with the higher utility. In the model of the cooperative game with transferable utility, we assume there is a period of preplay negotiation, during which the players meet to discuss the possibility of choosing a joint strategy together with some possible side payment to induce cooperation. They also discuss what will happen if they cannot come to an agreement; each may threaten to use some unilateral strategy that is bad for the opponent. If they do come to an agreement, it may be assumed that the payoff vector is Pareto optimal.

In the discussion, both players may make some threat of what strategy they will take if an agreement is not reached. However, a threat to be believable must not hurt the player who makes it to a greater degree than the opponent. Such a threat would not be credible. For example, consider the following bimatrix game.

$$\begin{pmatrix} (5,3) & (0,-4) \\ (0,0) & (3,6) \end{pmatrix}$$

If the players come to an agreement, it will be to use the lower right corner because it has the greatest total payoff, namely 9. Player II may argue that she should receive at least half the sum, 4.5. She may even feel generous in 'giving up' as a side payment some of the 6 she would be winning. However, Player I may threaten to use row 1 unless he is given at least 5. That threat is very credible since if Player I uses row 1, Player II cannot make a counter-threat to use column 2 because it would hurt her more than Player I. The counter-threat would not be credible.

In this model of the preplay negotiation, the threats and counter-threats may be made and remade until time to make a decision. Ultimately the players announce what threats they will carry out if agreement is not reached. It is assumed that if agreement is not reached, the players will leave the negotiation table and carry out their threats. However, being rational players, they will certainly reach agreement, since this gives a higher utility. The threats are only a formal method of arriving at a reasonable amount for the side payment, if any, from one player to the other.

The problem then is to choose the threats and the proposed side payment judiciously. The players use threats to influence the choice of the final payoff vector. The problem is how do the threats influence the final payoff vector, and how should the players choose their threat strategies? For two-person games with transferable utility, there is a very convincing answer.

Definition 3.4.1 (Threat solution)**.** *Let (A, B) be a game bimatrix.*

1. *The* **threat matrix** *is the matrix $T = A - B$.*
2. *The* **threat differential** *δ is the value of the threat matrix $T = A - B$. In other words, $\delta = v(T) = v(A - B)$.*
3. *The* **threat strategies** *of Player I and Player II are the maximin strategy $\mathbf{p}_d$ and the minimax strategy $\mathbf{q}_d$ of the threat matrix $T = A - B$ respectively.*
4. *The* **threat point***, or* **disagreement point***, is the payoff pair (μ_d, ν_d) when the threat strategies $\mathbf{p}_d$, $\mathbf{q}_d$ are being used. In other words,*
$$(\mu_d, \nu_d) = (\mathbf{p}_d A \mathbf{q}_d^T, \mathbf{p}_d B \mathbf{q}_d^T).$$
Note that $\delta = \mu_d - \nu_d$.
5. *The* **threat solution** *is the payoff pair*
$$(\varphi_1, \varphi_2) = \left(\frac{\sigma + \delta}{2}, \frac{\sigma - \delta}{2}\right) = \left(\frac{\sigma + \mu_d - \nu_d}{2}, \frac{\sigma - \mu_d + \nu_d}{2}\right).$$
where
$$\sigma = \max_{i,j}(a_{ij} + b_{ij})$$

is the **maximum total payoff** *which is the maximum entry of the sum matrix* $A+B$*. Note that* (φ_1, φ_2) *is the solution to*

$$\begin{cases} \varphi_1 + \varphi_2 = \sigma \\ \varphi_1 - \varphi_2 = \delta \end{cases}$$

If the players come to an agreement, then they will agree to play to achieve the largest possible total payoff $\sigma \max_{i,j}(a_{ij} + b_{ij})$ as the payoff to be divided between them. So it is easy to see that the threat solution (φ_1, φ_2) should satisfy $\varphi_1 + \varphi_2 = \sigma$.

Suppose now that the players have selected their threat strategies, $\mathbf{p}_d$ for Player I and $\mathbf{q}_d$ for Player II. Then if agreement is not reached, Player I receives $\mathbf{p}_d A \mathbf{q}_d^T$ and Player II receives $\mathbf{p}_d B \mathbf{q}_d^T$. The resulting payoff vector, $(\mu_d, \nu_d) = (\mathbf{p}_d A \mathbf{q}_d^T, \mathbf{p}_d B \mathbf{q}_d^T)$ is in the cooperative region and is called the disagreement point or threat point. Once the disagreement point is determined, the players must agree on the point (u, v) on the line $u + v = \sigma$ to be used as the cooperative solution. Player I will accept no less than μ_d and Player II will accept no less than ν_d since these can be achieved if no agreement is reached. But once the disagreement point has been determined, the game becomes symmetric. The players are arguing about which point on the line interval from $(\mu_d, \sigma - \mu_d)$ to $(\sigma - \nu_d, \nu_d)$ to select as the cooperative solution. No other considerations with respect to the matrices A and B play any further role. Therefore, the midpoint of the interval, namely

$$(\varphi_1, \varphi_2) = \left(\frac{\sigma + \mu_d - \nu_d}{2}, \frac{\sigma - \mu_d + \nu_d}{2} \right)$$

is the natural compromise. Both players suffer equally if the agreement is broken. Suppose Player I receives less than φ_1. He may threat Player II by saying that he will use his threat strategy $\mathbf{p}_d$. By doing so, Player I may guarantee that he gets at least $\delta = v(A - B)$ more than Player II. This ensures Player II will suffer more. Similarly, If Player II receives less than φ_2, she may ensure that Player I suffers more by using her threat strategy $\mathbf{q}_d$.

Example 3.4.2. *Find the threat strategies and the threat solution of the game bimatrix*

$$(A, B) = \begin{pmatrix} (0,0) & (6,2) & (-1,2) \\ (4,-1) & (3,6) & (5,5) \end{pmatrix}.$$

Solution. There is a Nash equilibrium in the first row, second column, with payoff vector $(6, 2)$. The maximum total payoff is

$$\sigma = 5 + 5 = 10.$$

If they come to an agreement, Player I will select the second row, Player II will select the third column and both players will receive a payoff of 5. They must still decide on a side payment, if any. They consider the zero-sum game with the threat matrix

$$T = A - B = \begin{pmatrix} 0 & 4 & -3 \\ 5 & -3 & 0 \end{pmatrix}.$$

The first column is strictly dominated by the last. The threat strategies are then easily determined to be

$$\begin{cases} \mathbf{p}_d = (0.3, 0.7) \\ \mathbf{q}_d = (0, 0.3, 0.7) \end{cases}$$

Now the threat differential is $\delta = v(A - B) = -9/10$ and the threat solution is

$$(\varphi_1, \varphi_2) = \left(\frac{10 - \frac{9}{10}}{2}, \frac{10 + \frac{9}{10}}{2} \right) = \left(\frac{91}{20}, \frac{109}{20} \right)$$

Example 3.4.3. *Find the threat solution of the game bimatrix*

$$(A, B) = \begin{pmatrix} (1,5) & (2,2) & (0,1) \\ (4,2) & (1,0) & (2,1) \\ (5,0) & (2,3) & (0,0) \end{pmatrix}.$$

Solution. There are two cooperative strategies giving total payoff $\sigma = 6$. The threat matrix is

$$T = A - B = \begin{pmatrix} -4 & 0 & -1 \\ 2 & 1 & 1 \\ 5 & -1 & 0 \end{pmatrix}.$$

which has a saddle-point at the $2, 3$-entry and the threat differential is $\delta = v(T) = 1$. The threat strategies are

$$\begin{cases} \mathbf{p}_d = (0, 1, 0) \\ \mathbf{q}_d = (0, 0, 1) \end{cases}$$

and the threat solution is

$$(\varphi_1, \varphi_2) = \left(\frac{6+1}{2}, \frac{6-1}{2}\right) = (3.5, 2.5).$$

Exercise 3

1. Find all Nash equilibria of the following bimatrix games. For each of the Nash equilibrium, find the payoff pair.

   (a) $\begin{pmatrix} (1,4) & (5,1) \\ (4,2) & (3,3) \end{pmatrix}$

   (b) $\begin{pmatrix} (5,2) & (2,0) \\ (1,1) & (3,4) \end{pmatrix}$

   (c) $\begin{pmatrix} (1,5) & (2,3) \\ (5,2) & (4,2) \end{pmatrix}$

   (d) $\begin{pmatrix} (-1,0) & (2,1) \\ (4,3) & (-3,-1) \end{pmatrix}$

2. Find all Nash equilibria of the following bimatirx games

   (a) $\begin{pmatrix} (4,1) & (2,3) & (3,4) \\ (3,2) & (5,5) & (1,2) \end{pmatrix}$

   (b) $\begin{pmatrix} (1,0) & (4,-1) & (5,1) \\ (3,2) & (1,1) & (2,-1) \end{pmatrix}$

   (c) $\begin{pmatrix} (4,6) & (0,3) & (2,-1) \\ (2,4) & (6,5) & (-1,1) \\ (5,0) & (1,2) & (4,3) \end{pmatrix}$

   (d) $\begin{pmatrix} (3,2) & (4,0) & (7,9) \\ (2,6) & (8,4) & (3,5) \\ (5,4) & (5,3) & (4,1) \end{pmatrix}$

3. The Brouwer's fixed-point theorem states that every continuous map $f : X \to X$ has a fixed-point if X is homeomorphic to a closed unit ball. Find a map $f : X \to X$ which does not have any fixed-point for each of the following topological spaces X. (It follows that the following spaces are not homeomorphic to a closed unit ball.)

   (a) X is the punched closed unit disc $D^2 \setminus \{0\} = \{(x,y) \in \mathbb{R}^2 : 0 < x^2 + y^2 \leq 1\}$

   (b) X is the unit sphere $S^2 = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$

   (c) X is the open unit disc $B^2 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$

4. For each of the following bimatrices (A, B), find the values ν_A and ν_{B^T} of A and B^T respectively, and the Nash bargaining solution using $(\mu, \nu) = (\nu_A, \nu_{B^T})$ as the status quo point.

(a) $\begin{pmatrix} (4,-4) & (-1,-1) \\ (0,1) & (1,0) \end{pmatrix}$ (c) $\begin{pmatrix} (2,2) & (0,1) & (1,-1) \\ (4,1) & (-2,1) & (1,3) \end{pmatrix}$

(b) $\begin{pmatrix} (3,1) & (1,0) \\ (0,-1) & (2,3) \end{pmatrix}$ (d) $\begin{pmatrix} (6,4) & (0,10) & (4,1) \\ (8,-2) & (4,1) & (0,1) \end{pmatrix}$

5. Two broadcasting companies, NTV and CTV, bid for the exclusive broadcasting rights of an annual sports event. If both companies bid, NTV will win the bidding with a profit of \$20 (million) and CTV will have no profit. If only NTV bids, there'll be a profit of \$50 (million). If only CTV bids, there'll be a profit of \$40 (million). Find the Nash's solution to the bargaining problem.

6. Let $\mathcal{R} = \{(u,v) : v \geq 0 \text{ and } u^2 + v \leq 4\} \subset \mathbb{R}^2$. Find the arbitration pair $A(\mathcal{R}, (\mu,\nu))$ using the following points as the status quo point (μ,ν).

   (a) $(0,0)$ (b) $(0,1)$

7. Let $\mathcal{R} \subset \mathbb{R}^2$ be a closed and bounded convex set, $(\mu,\nu) \in \mathcal{R}$ and $(\alpha,\beta) = A(\mathcal{R},(\mu,\nu))$ be the arbitration pair with $\alpha \neq \mu$. Suppose the boundary of $\mathcal{R}$ is given, locally at (α,β), by the graph of a differentiable function $f(x)$ with $f(\alpha) = \beta$. Prove that $f'(\alpha)$ is equal to the negative of the slope of the line joining (μ,ν) and (α,β).

8. Suppose A is an $n \times n$ matrix such that the sum of entries in any row of A is equal to a constant rn. Let (μ,ν) be the status quo point of the bimatrix (A, A^T).

   (a) Prove that there is a Nash equilibrium of (A, A^T) with (r,r) as payoff pair.

   (b) Prove that the arbitration payoff pair of the bimatrix (A, A^T) is $(\alpha,\beta) = (m,m)$ where m is the maximum entry of $\dfrac{A + A^T}{2}$. (Here in finding the arbitration payoff pair of bimatix (A,B), the status quo point is taken to be $(\mu,\nu) = (v(A), v(B^T))$ where v is the value of a matrix.)

9. Find the threat strategies and the threat solutions of the following game bimatrix.

(a) $\begin{pmatrix} (3,-2) & (2,4) \\ (1,0) & (3,-1) \end{pmatrix}$

(b) $\begin{pmatrix} (5,3) & (1,3) \\ (4,4) & (2,1) \end{pmatrix}$

(c) $\begin{pmatrix} (6,4) & (2,3) & (4,7) \\ (2,6) & (4,2) & (5,4) \end{pmatrix}$

(d) $\begin{pmatrix} (2,8) & (7,5) & (6,3) \\ (0,7) & (4,3) & (5,5) \\ (3,-1) & (-2,6) & (2,7) \end{pmatrix}$

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