MMAT5320 Computational Mathematics

Take Home Final Exam

- There are a total of 3 questions, and the total score is **126** points.
- This is to make up from the abrupt end of the semester. There will be no questions on the material in the last chapter on Eigenvalue algorithms.
- Please write down your answers in **pen** (no pencil work will be accepted), then provide a scan of your answers and email them to me at kflam@math.cuhk.edu.hk
- You must send your answers via email by **20th December 2019** 23:59pm.
- Late answers will not be accepted! There will be **no** exceptions.

Question 1

(a) [8 pts] Let $A \in \mathbb{R}^{m\times n}$ be a matrix of full rank. Write down the procedure to obtain a reduced SVD of A.

(b) [10 pts] Compute a reduced SVD of the following matrix

$$A = \begin{pmatrix} 2 & 0 \\ 1 & 2 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

(c) [8 pts] Find two more vectors $\vec{u}_3$ and $\vec{u}_4$ in $\mathbb{R}^4$ such that the set $\{\vec{u}_1, \dots, \vec{u}_4\}$ with the following $\vec{u}_1$ and $\vec{u}_2$ forms an orthonormal set:

$$\vec{u}_1 = \frac{1}{\sqrt{14}} \begin{pmatrix} 2 \\ 3 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{u}_2 = \frac{1}{\sqrt{6}} \begin{pmatrix} 2 \\ -1 \\ -1 \\ 0 \end{pmatrix}.$$

In your answer demonstrate clearly that $\{\vec{u}_1, \dots, \vec{u}_4\}$ satisfies all the requirements to be an orthonormal set.

(d) [3 pts] Write down a full SVD of the matrix A in part (b).

Question 2

(a) [20 pts] Let $\{\vec{a}_1, \dots, \vec{a}_n\}$, where $\vec{a}_i \in \mathbb{R}^m$ for $1 \le i \le n$ and $m > n$, be a set of linearly independent vectors. Describe one method to derive a full QR factorisation of the matrix A whose columns are $\{\vec{a}_1, \dots, \vec{a}_n\}$. You may choose from the standard Gram–Schmidt orthonormalisation procedure, Gram–Schmidt projections or Householder reflections.

In your answer, outline the details for step 1, step 2 and for step k, and show how to construct the matrices Q and R.

(b) [13 pts] Compute the reduced QR factorisation of the following matrix

$$B = \begin{pmatrix} 1 & 5 & 2 \\ 1 & -1 & 2 \\ 1 & 5 & 4 \\ 1 & -1 & 4 \end{pmatrix}.$$

You may use any procedure for your calculations (Gram–Schmidt, Gram–Schmidt projections, Householder reflections).

(c) [6 pts] Compute the pseudoinverse corresponding to B, and compute the least squares solution $\vec{x}_* \in \mathbb{R}^3$ to the problem $\|B\vec{x}_* - \vec{b}\|_2 \le \|B\vec{y} - \vec{b}\|_2$ for any $\vec{y} \in \mathbb{R}^3$ if $\vec{b} = (1, 1, -1, -1)^\top$.

(d) [9 pts] Let $\vec{x}_1$ and $\vec{x}_2$ be two linearly independent vectors in $\mathbb{R}^3$, and let $P = \operatorname{span}\{\vec{x}_1, \vec{x}_2\}$ denote the plane spanned by these two vectors. Let $X \in \mathbb{R}^{3\times 2}$ be the matrix whose columns are $\vec{x}_1$ and $\vec{x}_2$, and let $X = QR$ be a full QR factorisation.

- Show that the first two columns of Q, denoted by $\vec{q}_1$ and $\vec{q}_2$, lie in the plane P.
- Use the full QR factorisation to find a normal direction to the plane P.

Question 3

(a) [20 pts]

- Given an example of a 3×3 matrix which is not diagonalisable.
- Given an example of a 2×2 matrix which is diagonalisable, but not unitary diagonalisable.

In your answer you must demonstrate why your chosen matrix satisfies the required properties.

(b) [20 pts] Compute the eigenvalues and the Schur factorisation of the following matrix

$$A = \begin{pmatrix} 2 & 0 & 0 & 0 \\ -1 & 3 & 0 & 0 \\ 0 & 0 & 6 & 7 \\ 0 & 0 & 0 & 8 \end{pmatrix}$$

(c) [9 pts] For a general square matrix $A \in \mathbb{R}^{m\times m}$, let $A = QTQ^\top$ denote a Schur factorisation with orthogonal Q and upper triangular T, where $\vec{q}_i$, $1 \le i \le m$ denotes the ith column of Q. Show that the second column $\vec{q}_2$ is an eigenvector to the matrix $B := (I - \vec{q}_1\vec{q}_1^\top)A(I - \vec{q}_1\vec{q}_1^\top)$ and compute the corresponding eigenvalue.