MMAT5320 Computational Mathematics

Assignment 1

Due date: 10th October 2019

Please hand in your assignments to the assignment box on 2/F Lady Shaw Building (opposite the administration office and underneath the notice boards) by 6pm on **Thursday 10th October 2019**. Remember to include your name, ID number and show all of your working!

Question 1

Calculate the rank, range and nullspace of the following matrices:

$$A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 7 & 0 & 1 \\ 3 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad A_3 = \begin{pmatrix} a_1a_1 & a_1a_2 & a_1a_3 \\ a_2a_1 & a_2a_2 & a_2a_3 \\ a_3a_1 & a_3a_2 & a_3a_3 \end{pmatrix}$$

where a_1, a_2, a_3 are non-zero real numbers.

0.1 Solutions

The rank, range and nullspace of A_1 are [1 point each]

$$\text{rank}(A_1) = 1, \quad \text{range}(A_1) = \{(x,0,0) : x \in \mathbb{R}\}, \quad \text{null}(A_1) = \{(0,y,z) : y,z \in \mathbb{R}\}.$$

For A_2 [1 point each]

$$\text{rank}(A_2) = 2, \quad \text{range}(A_2) = \{(7x+z, 3x, 2z) : x,z \in \mathbb{R}\}, \quad \text{null}(A_2) = \{(0,y,0) : y \in \mathbb{R}\}.$$

For A_3 [1 point for rank, 2 points each for range and nullspace]

$$\text{rank}(A_3) = 1, \quad \text{range}(A_3) = \text{span}\{(a_1,a_2,a_3)\}, \quad \text{null}(A_3) = \text{span}\{(\frac{1}{a_1}, 0, -\frac{1}{a_3}), (0, \frac{1}{a_2}, -\frac{1}{a_3})\}.$$

[Total 11 points]

Question 2

(i) Fix any two real numbers $p < q$, prove that for any $x \in \mathbb{R}^m$, $m \geq 1$, the following inequality for vector norms holds

$$\|x\|_q \leq \|x\|_p.$$

Hint: you can use without proof the following property: if $(a_i)_{i=1,\dots,m}$ are non-negative constants and $s \geq 1$, then

$$\sum_{i=1}^m a_i^s \leq \Big(\sum_{i=1}^m a_i\Big)^s.$$

(ii) Give a counterexample to the claim

$$\|x\|_2 > \|x\|_1.$$

What modification to the left-hand side is needed to make the inequality true?

(iii) For a square matrix $A \in \mathbb{R}^{m\times m}$, we define a function $\|A\|_* := \max_{1\le i,j\le n} |a_{ij}|$. Show that this is a norm.

(iv) Give an example of a matrix $A \in \mathbb{R}^{2\times 2}$ where $\|A^2\|_* > \|A\|_*^2$, and explain why this shows that $\|\cdot\|_*$ is not an induced p-matrix norm.

(v) For the matrix

$$A = \begin{pmatrix} 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

calculate $\|A\|_2$ and $\|A\|_4$.

0.2 Solutions

(i) Set $y = \frac{x}{\|x\|_p}$, then $\|y\|_p = 1$, which means $\|y\|_p^p = \sum_{i=1}^m |y_i|^p = 1$ [1 point]. Moreover,

$$\sum_{i=1}^m |y_i|^q = \sum_{i=1}^m |y_i|^{p\frac{q}{p}} \le \Big(\sum_{i=1}^m |y_i|^p\Big)^{\frac{q}{p}} = 1 \quad [\text{ 3 points: 1 point each for } =, \le, =]$$

since $q/p > 1$. This shows $\|y\|_q \le 1$, and so

$$\|y\|_q = \frac{\|x\|_q}{\|x\|_p} \le 1 \qquad \Longrightarrow \qquad \|x\|_q \le \|x\|_p.$$

[2 points: 1 for $\|y\|_q \le 1$ and 1 for conclusion].

(ii) Counterexample (any would suffices), e.g. $x = (1, 1, 0)$. [2 full points awarded to calculations showing $\|x\|_2 \le \|x\|_1$, otherwise just 1 point for stating the counter example]

Modifications would be $m\|x\|_2 > \|x\|_1$ for $x \in \mathbb{R}^m$. [1 point]

(iii) [1 point each for the three requirements of a norm, total 3 points]

(iv) Example

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

then calculations show [2 points: 1 point each for computing matrix norms] $\|A^2\|_* = 2 > \|A\|_*^2 = 1$.

If $\|\cdot\|_*$ is a induced p-matrix norm, then from Lemma in Lecture notes slide (Inequalities I) we must have

$$\|A^2\|_* \le \|A\|_*^2,$$

but the counterexample shows this cannot be true. [2 points: 1 for applying Lemma, 1 for deducing contradiction]

(v) For $x \in \mathbb{R}^2$, $Ax = (2x_2, 0, 0)$. Then,

$$\|Ax\|_2 = 2|x_2|, \quad \|Ax\|_4 = 2|x_2|. \quad \text{[2 points: 1 for each]}$$

So

$$\|A\|_2 = \sup_{x_1^2+x_2^2=1} 2|x_2| = 2, \quad \|A\|_4 = \sup_{x_1^4+x_2^4=1} 2|x_2| = 2 \quad \text{[2 points: 1 for each]}$$

[Total 20 points]

Question 3

(i) Compute a full SVD for the following matrices:

$$A_1 = \begin{pmatrix} 2 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & i \\ 0 & 0 \end{pmatrix}$$

where $i = \sqrt{-1}$ is the imaginary unit. Also write down a reduced SVD for each matrix.

(ii) Let $A \in \mathbb{C}^{m\times m}$ be a hermitian and unitary matrix, with $A = U\Sigma V^*$ be a SVD of A. Show that

- all eigenvalues of A are 1 or -1;
- all singular values of A are 1.

0.3 Solutions

(i) Compute [1 point]

$$A_1^\top A_1 = \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}$$

Eigenvalues are roots of characteristic polynomial $x^2 - 5x = 0$ implies $\sigma_1^2 = 5$ and $\sigma_2^2 = 0$ [2 points]. Eigenvectors for σ_1^2 and σ_2^2 with unit length are [2 points]

$$v_1 = \begin{pmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix} \quad v_2 = \begin{pmatrix} 1/\sqrt{5} \\ -2/\sqrt{5} \end{pmatrix}$$

Then, $u_1 = \frac{1}{\sigma_1} A_1 v_1 = 1$ [1 point], and the full SVD is [1 point]

$$A_1 = \underbrace{1}_{U} \underbrace{\begin{pmatrix} \sqrt{5} & 0 \end{pmatrix}}_{\Sigma} \underbrace{\frac{1}{\sqrt{5}} \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}}_{V^\top}.$$

A reduced SVD is [1 point]

$$A_1 = \underbrace{\begin{pmatrix} 1 & 0 \end{pmatrix}}_{\hat{U}} \underbrace{\begin{pmatrix} \sqrt{5} & 0 \\ 0 & 0 \end{pmatrix}}_{\hat{\Sigma}} \underbrace{\frac{1}{\sqrt{5}} \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}}_{V^\top}.$$

For A_2, compute [1 point]

$$A_2^* A_2 = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$$

Eigenvalues are roots of characteristic polynomial $x^2 - 2x = 0$ implies $\sigma_1^2 = 2$ and $\sigma_2^2 = 0$ [2 points]. Eigenvectors with unit length are [2 points]

$$v_1 = \begin{pmatrix} 1/\sqrt{2} \\ -i/\sqrt{2} \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1/\sqrt{2} \\ i/\sqrt{2} \end{pmatrix}$$

Then,

$$u_1 = \frac{1}{\sigma_1} A v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and we take $u_2 = (0,1)^\top$ [2 points]. Then, the full SVD is [1 point]

$$A_2 = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{U} \underbrace{\begin{pmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{pmatrix}}_{\Sigma} \underbrace{\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}}_{V^*}.$$

The reduced SVD is equal to the full SVD since the matrix is a square matrix [1 point]

(ii) If A is hermitian and unitary, then $A^* = A$ and $A^* A = A^2 = I$ [2 points]. If $A = U\Sigma V^*$ is a SVD of A, then

$$I = A^* A = V\Sigma^* U^* U \Sigma V^* = V\Sigma^*\Sigma V^*. \quad \text{[2 points]}$$

Since Σ is a real diagonal matrix, $\Sigma^*\Sigma = \Sigma^2$ which contains the singular values of A. Therefore,

$$I = V^* V = V^*(V\Sigma^2 V^*)V = \Sigma^2,$$

and so all singular values of A are 1. [1 point]

If λ is an eigenvalue of A with corresponding eigenvector x, then

$$x = A^* A x = \lambda A^* x = \lambda A x = \lambda^2 x \quad \text{[2 points]}$$

and so $\lambda^2 = 1$, which implies $\lambda = \pm 1$. [1 point]

[Total 25 points]

Question 4

(i) For the matrix

$$A = \begin{pmatrix} 2 & 1 \\ 0 & 1 \\ 2 & 0 \end{pmatrix}$$

find the orthogonal projector P onto range(A). What is the projection of a point $(1,2,3)^\top$ to range(A)?

(ii) Consider the matrix

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 4 & 8 & -4 \end{pmatrix}.$$

- Find the orthogonal projector P_C onto range(A).
- Find the orthogonal projector P_R onto range$(A^\top)$.
- Prove that the following relation holds for general matrices $A \in \mathbb{R}^{m\times n}$

$$P_C A P_R = A.$$

0.4 Solutions

(i) Orthogonal projector to range(A) is [3 points]

$$P = A(A^\top A)^{-1}A^\top = \begin{pmatrix} 2 & 1 \\ 0 & 1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 1/6 & -1/6 \\ -1/6 & 2/3 \end{pmatrix} \begin{pmatrix} 2 & 0 & 2 \\ 1 & 1 & 0 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$$

Projection of point $(1,2,3)^\top$ is [1 point]

$$P \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 7 \\ 2 \\ 5 \end{pmatrix}.$$

(ii) Note that the columns 2 and 3 are linear combinations of column 1, and so range$(A) =$ span$\{(1,4)^\top\}$ [1 point]. The orthogonal projector onto range(A) is therefore [2 points]

$$P_C = \frac{1}{\|c_1\|_2^2}(c_1 c_1^\top) = \frac{1}{17} \begin{pmatrix} 1 & 4 \\ 4 & 16 \end{pmatrix}.$$

Meanwhile, row 2 is a linear combination of row 1, and so range$(A^\top) =$ span$\{(1,2,-1)\}$ [1 point]. The orthogonal projector onto range$(A^\top)$ is therefore [2 points]

$$P_R = \frac{1}{\|r_1\|_2^2} r_1 r_1^\top = \frac{1}{6} \begin{pmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \\ -1 & -2 & 1 \end{pmatrix}.$$

For general matrices $A \in \mathbb{R}^{m\times n}$ and any vector $v \in \mathbb{R}^n$, we see that $Av \in$ range(A) and so $P_C Av = Av$, i.e., $P_C A = A$ [1 point]. Since P_R is the orthogonal projector to range$(A^\top)$, for any pair of vectors w and v, it holds

$$\begin{aligned} (AP_R v)^\top w &= (AP_R^\top v)^\top w \text{ [hermitian/symmetric of projector]} \\ &= v^\top P_R A^\top w = v^\top A^\top w = (Av)^\top w. \end{aligned}$$

This implies $AP_R = A$ [2 point], and so $P_C A P_R = P_C A = A$ [1 point].

[Total 14 points]