MMAT 5220 Complex Analysis and Its Applications

Lecture 6

§ Series

Thm Suppose that a function f is analytic throughout an annulus $A := \{z \in \mathbb{C} : R_1 < |z - z_0| < R_2\}$ $(0 \le R_1 < R_2 \le +\infty)$ centered at z_0.

Let γ be a positively oriented simple closed contour in A.

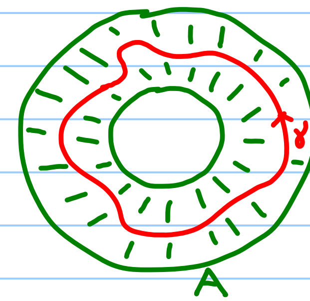
Then $\forall z \in A$, $f(z)$ has a Laurent series representation:

$$(*) \quad f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n \text{ in } A, \text{ where } a_n = \frac{1}{2\pi i} \int_\gamma \frac{f(s)}{(s - z_0)^{n+1}} ds \quad \forall n \in \mathbb{Z}.$$

Furthermore, if f is actually analytic in $B(z_0, R_2) = \{z \in \mathbb{C} : |z - z_0| < R_2\}$, then $a_n = 0$ $\forall n < 0$ and $f(z)$ has a Taylor series representation:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \text{ in } B(z_0, R_2).$$

Rmk (*) means both infinite series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ and $\sum_{k=1}^{\infty} a_{-k}(z-z_0)^{-k}$ converge and their sum is equal to $f(z)$ $\forall z \in A$.



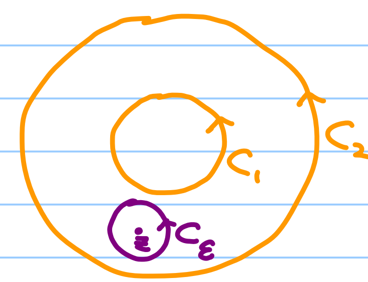
Pf: WLOG, we can assume that $z_0 = 0$.

Let $z \in A$. We can choose r_1, r_2 s.t. $R_1 < r_1 < |z| < r_2 < R_2$.

Then f is analytic on the circles $C_i := \{z \in \mathbb{C} : |z| = r_i\}$, and the annulus $A' := \{z \in \mathbb{C} : r_1 < |z| < r_2\}$. Choose $\varepsilon > 0$ s.t. $B(z,\varepsilon) \subset A'$.

Apply Cauchy-Goursat Thm to $\frac{f(s)}{s-z}$ gives

$$\int_{C_2} \frac{f(s)}{s-z}\,ds - \int_{C_1} \frac{f(s)}{s-z}\,ds - \int_{C_\varepsilon} \frac{f(s)}{s-z}\,ds = 0$$



Then by Cauchy integral formula, we have

$$f(z) = \frac{1}{2\pi i}\int_{C_2} \frac{f(s)}{s-z}\,ds = \frac{1}{2\pi i}\int_{C_2} \frac{f(s)}{s-z}\,ds - \frac{1}{2\pi i}\int_{C_1} \frac{f(s)}{s-z}\,ds.$$

For $s \in C_2$, we have $\left|\frac{z}{s}\right| = \frac{|z|}{r_2} < 1$ and

$$\begin{aligned}
\frac{1}{2\pi i}\int_{C_2} \frac{f(s)}{s-z}\,ds &= \frac{1}{2\pi i}\int_{C_2} \frac{f(s)}{s}\,\frac{1}{1-\frac{z}{s}}\,ds \\
&= \frac{1}{2\pi i}\int_{C_2} \frac{f(s)}{s}\left[\sum_{n=0}^{N-1}\left(\frac{z}{s}\right)^n + \frac{\left(\frac{z}{s}\right)^N}{1-\frac{z}{s}}\right]ds \\
&= \sum_{n=0}^{N-1}\left(\frac{1}{2\pi i}\int_{C_2} \frac{f(s)}{s^{n+1}}\,ds\right) z^n + \frac{1}{2\pi i}\int_{C_2} \frac{f(s)}{s-z}\left(\frac{z}{s}\right)^N ds
\end{aligned}$$

Now letting $M_2 = \sup_{s \in C_2} |f(s)|$ and $r := |z|$, we have

$$\left|\frac{1}{2\pi i}\int_{C_2} \frac{f(s)}{s-z}\left(\frac{z}{s}\right)^N ds\right| \le \frac{1}{2\pi}\cdot\frac{M_2}{r_2 - r}\left(\frac{r}{r_2}\right)^N \cdot 2\pi r_2 = \frac{M_2 r_2}{r_2 - r}\left(\frac{r}{r_2}\right)^N \to 0 \quad \text{as } N \to \infty$$

So $\frac{1}{2\pi i}\int_{C_2} \frac{f(s)}{s-z}\,ds = \sum_{n=0}^{\infty} a_n z^n$

where $a_n = \frac{1}{2\pi i}\int_{C_2} \frac{f(s)}{s^{n+1}}\,ds = \frac{1}{2\pi i}\int_{\gamma} \frac{f(s)}{s^{n+1}}\,ds$ by independence of paths.

For $s \in C_1$, we have $\left|\frac{z}{s}\right| = \frac{|z|}{r_1} > 1$ and

$$-\frac{1}{2\pi i}\int_{C_1} \frac{f(s)}{s-z}\,ds = \frac{1}{2\pi i}\int_{C_1} \frac{f(s)}{z}\,\frac{1}{1-\frac{s}{z}}\,ds$$

$$= \frac{1}{2\pi i}\int_{C_1} \frac{f(s)}{z}\left[\sum_{n=0}^{N-1}\left(\frac{s}{z}\right)^n + \frac{\left(\frac{s}{z}\right)^N}{1-\frac{s}{z}}\right]ds$$

$$= \sum_{n=0}^{N-1}\left(\frac{1}{2\pi i}\int_{C_1} f(s)\,s^n\,ds\right)\frac{1}{z^{n+1}} + \frac{1}{2\pi i}\int_{C_1} \frac{f(s)}{z-s}\left(\frac{s}{z}\right)^N ds$$

$$= \sum_{n=1}^{N}\left(\frac{1}{2\pi i}\int_{C_1} f(s)\,s^{n-1}\,ds\right)\frac{1}{z^n} + \frac{1}{2\pi i}\int_{C_1} \frac{f(s)}{z-s}\left(\frac{s}{z}\right)^N ds$$

Now letting $M_1 = \sup_{s \in C_1} |f(s)|$, we have

$$\left| \frac{1}{2\pi i} \int_{C_1} \frac{f(s)}{z-s} \left(\frac{s}{z}\right)^N ds \right| \leq \frac{1}{2\pi} \cdot \frac{M_1}{r-r_1} \left(\frac{r_1}{r}\right)^N \cdot 2\pi r_1 = \frac{M_1 r_1}{r-r_1} \left(\frac{r_1}{r}\right)^N \to 0 \quad \text{as } N \to \infty$$

So $-\frac{1}{2\pi i} \int_{C_1} \frac{f(s)}{s-z} ds = \sum_{n=1}^{\infty} a_{-n} z^{-n}$

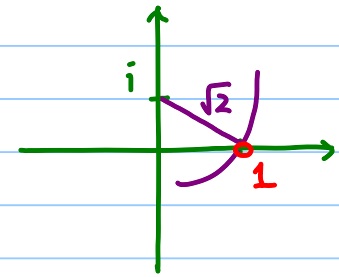
where $a_{-n} = \frac{1}{2\pi i} \int_{C_1} \frac{f(s)}{s^{-n+1}} ds = \frac{1}{2\pi i} \int_{\gamma} \frac{f(s)}{s^{-n+1}} ds$ by independence of paths.

Finally, if f is actually analytic in $B(0, R_2)$, then $\forall n \geq 1$, $f(s) s^{n-1}$ is analytic inside and on C_1, so $a_{-n} = 0$. #

e.g.

- $\dfrac{1}{1-z} = \sum_{n=0}^{\infty} z^n = 1 + z + z^2 + \cdots$ for $|z|<1$
- $e^z = \sum_{n=0}^{\infty} \dfrac{z^n}{n!} = 1 + z + \dfrac{z^2}{2!} + \dfrac{z^3}{3!} + \cdots$ for $z \in \mathbb{C}$
- $\sin z = \sum_{n=0}^{\infty} \dfrac{(-1)^n}{(2n+1)!} z^{2n+1} = z - \dfrac{z^3}{3!} + \dfrac{z^5}{5!} + \cdots$ for $z \in \mathbb{C}$
- $\cos z = \sum_{n=0}^{\infty} \dfrac{(-1)^n}{(2n)!} z^{2n} = 1 - \dfrac{z^2}{2!} + \dfrac{z^4}{4!} + \cdots$ for $z \in \mathbb{C}$
- $\sinh z = \sum_{n=0}^{\infty} \dfrac{1}{(2n+1)!} z^{2n+1} = z + \dfrac{z^3}{3!} + \dfrac{z^5}{5!} + \cdots$ for $z \in \mathbb{C}$
- $\cosh z = \sum_{n=0}^{\infty} \dfrac{1}{(2n)!} z^{2n} = 1 + \dfrac{z^2}{2!} + \dfrac{z^4}{4!} + \cdots$ for $z \in \mathbb{C}$
- $\mathrm{Log}(1+z) = \sum_{n=0}^{\infty} \dfrac{(-1)^n}{n+1} z^{n+1} = z - \dfrac{z^2}{2} + \dfrac{z^3}{3} + \cdots$ for $|z|<1$

e.g. $f(z) = \frac{1}{1-z}$ is analytic at $z_0 = i$.



In fact, $f(z)$ is analytic in $|z-i| < \sqrt{2}$

To find its Taylor series, we compute

$$f(z) = \frac{1}{1-z} = \frac{1}{(1-i)-(z-i)} = \frac{1}{1-i} \cdot \frac{1}{1 - \frac{z-i}{1-i}}$$

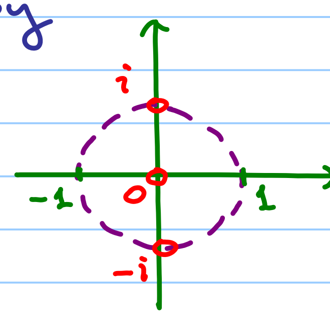
$$= \frac{1}{1-i} \sum_{n=0}^{\infty} \left(\frac{z-i}{1-i} \right)^n \qquad \left(\because \left| \frac{z-i}{1-i} \right| = \frac{|z-i|}{\sqrt{2}} < 1 \right)$$

$$= \sum_{n=0}^{\infty} \frac{1}{(1-i)^{n+1}} (z-i)^n$$

e.g. $\quad f(z) = \dfrac{1}{z(1+z^2)}$

For $0<|z|<1$, the Laurent expansion of f is given by

$$f(z) = \frac{1}{z}\,\frac{1}{1-(-z^2)} = \frac{1}{z}\sum_{n=0}^{\infty}(-z^2)^n = \sum_{n=0}^{\infty}(-1)^n z^{2n-1}$$



While for $|z|>1$, letting $w=\frac{1}{z}$, we have

$$f(z) = \frac{w^3}{1+w^2} = w^3\sum_{n=0}^{\infty}(-w^2)^n = \sum_{n=0}^{\infty}(-1)^n z^{-(2n+3)}$$

e.g. $\quad f(z) = e^{\frac{1}{z^2}}$ is analytic everywhere on $\mathbb{C}$ except at the origin.

Over $\mathbb{C}\setminus\{0\}$, the Laurent series expansion of f is given by

$$f(z) = \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{1}{z^2}\right)^n = \sum_{n=0}^{\infty}\frac{1}{n!\,z^{2n}} = 1+\frac{1}{z^2}+\frac{1}{2z^4}+\frac{1}{3!\,z^6}+\cdots$$

§ Power series

Def Given $z_0 \in \mathbb{C}$ and a power series $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ (where $a_n \in \mathbb{C}\ \forall n$), the greatest $R>0$ s.t. $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ converges for $|z-z_0|<R$ is called the radius of convergence and $\{z\in\mathbb{C} : |z-z_0|=R\}$ is called the circle of convergence of the series.

Rmk R could be $+\infty$.

Lemma If $\sum_{n=0}^{\infty} a_n(z_1 - z_0)^n$ converges for $z_1 \neq z_0$, then $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ is absolutely convergent for $|z - z_0| < |z_1 - z_0|$.

Rmk : $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ is absolutely convergent means that $\sum_{n=0}^{\infty} |a_n(z-z_0)^n|$ converges.

Pf: $\sum_{n=0}^{\infty} a_n(z_1 - z_0)^n$ converges $\Rightarrow \exists M > 0$ s.t. $|a_n(z_1-z_0)^n| \leq M \quad \forall n \in \mathbb{N}$.

Hence, for $|z - z_0| < |z_1 - z_0|$, we have

$$\sum_{n=0}^{\infty} |a_n(z-z_0)^n| = \sum_{n=0}^{\infty} |a_n(z_1-z_0)^n| \left|\frac{z-z_0}{z_1-z_0}\right|^n \leq M \sum_{n=0}^{\infty} r^n,$$

where $r = \frac{|z-z_0|}{|z_1-z_0|} < 1$. So $\sum_{n=0}^{\infty} |a_n(z-z_0)^n|$ converges by comparison test. #