MMAT 5220 Complex Analysis and Its Applications

Lecture 3

§ Cauchy Riemann equations (cont'd)

Let $f(z) = u(x,y) + i v(x,y)$. Then recall that $f'(z_0)$ exists at $z_0 = x_0 + iy_0$ if and only if the map $F : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $(x,y) \mapsto (u(x,y), v(x,y))$ is differentiable at (x_0,y_0) and satisfies the Cauchy-Riemann eqns :

$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases} \quad \text{at } (x_0,y_0);$$

in this case, $f'(z_0) = u_x(x_0,y_0) + i v_x(x_0,y_0)$.

Or if $f(z)$ is defined in some nbh $U \ni z_0$, and u_x, u_y, v_x, v_y exist in U and are continuous and satisfy the CR eqns at (x_0,y_0), then $f'(z_0)$ exists.

Now if we write z in polar coordinates : $z = re^{i\theta} = r(\cos\theta + i\sin\theta)$,

then $\begin{pmatrix} u_r \\ u_\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{pmatrix} \begin{pmatrix} u_x \\ u_y \end{pmatrix}$, $\begin{pmatrix} v_r \\ v_\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{pmatrix} \begin{pmatrix} v_x \\ v_y \end{pmatrix}$

So the CR eqns $\begin{pmatrix} u_x \\ u_y \end{pmatrix} = \begin{pmatrix} v_y \\ -v_x \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} v_x \\ v_y \end{pmatrix}$ become

$$A^{-1} \begin{pmatrix} u_r \\ u_\theta \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} A^{-1} \begin{pmatrix} v_r \\ v_\theta \end{pmatrix}$$ where $A = \begin{pmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{pmatrix}$

$$\Leftrightarrow \begin{pmatrix} u_r \\ u_\theta \end{pmatrix} = A \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} A^{-1} \begin{pmatrix} v_r \\ v_\theta \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \frac{1}{r} \\ -r & 0 \end{pmatrix} \begin{pmatrix} v_r \\ v_\theta \end{pmatrix} = \begin{pmatrix} \frac{1}{r} v_\theta \\ -r v_r \end{pmatrix}$$

(Note $A^{-1} = \begin{pmatrix} \cos\theta & -\frac{1}{r}\sin\theta \\ \sin\theta & \frac{1}{r}\cos\theta \end{pmatrix}$)

i.e. $\begin{cases} u_r = \frac{1}{r} v_\theta \\ u_\theta = -r v_r \end{cases}$ (CR eqns in polar coordinates)

e.g. $\mathrm{Log}: \mathbb{C} \setminus (-\infty, 0] \to \mathbb{C}$ is given by $\mathrm{Log}\, z = \log r + i\theta$

$$\begin{cases} u(r,\theta) = \log r \\ v(r,\theta) = \theta \end{cases} \Rightarrow \begin{cases} u_r = \frac{1}{r} = \frac{1}{r} v_\theta \\ u_\theta = 0 = -r v_r \end{cases}$$

So Log is differentiable.

Def • A function f is analytic in an open set U if $f'(z)$ exists $\forall z \in U$.

• If S is not open, then we say f is analytic in S if f is analytic in some open set U containing S. In particular, f is analytic at a point z_0 if f is analytic in $B(z_0, \rho)$ for some $\rho > 0$.

e.g. $f(z) = |z|^2$ is differentiable but not analytic at $z = 0$.

<u>Simple properties</u> :

• f, g analytic $\Rightarrow$ $f \pm g$, fg, f/g (provided that $g \neq 0$) are analytic

• f, g analytic $\Rightarrow$ $f \circ g$ analytic and $(f \circ g)'(z) = f'(g(z)) \cdot g'(z)$

- If f is analytic and $f'(z) = 0$ throughout a domain D, then $f \equiv$ const in D.
- If both f and $\bar{f}$ are analytic throughout a domain D, then $f \equiv$ const in D.
- If f is analytic and $|f| \equiv$ const throughout a domain D, then $f \equiv$ const in D.

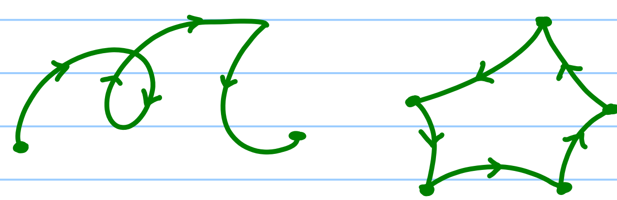
§ Contour integrals

Some terminologies

- An arc in $\mathbb{C}$ is a continuous function $\gamma : I \longrightarrow \mathbb{C}$ where $I = [a,b]$ is an interval in $\mathbb{R}$. We usually write γ as $\gamma(t) = x(t) + iy(t)$.
- We say that γ is closed if $\gamma(a) = \gamma(b)$, simple if $\gamma(t_1) \neq \gamma(t_2)$ $\forall\, t_1 \neq t_2$ and smooth if γ is continuously differentiable with $\gamma'(t) \neq 0$ over (a,b).
- A contour is a piecewise smooth arc $\gamma : [a,b] \longrightarrow \mathbb{C}$

  i.e. $\exists\; a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b$

  s.t. $\gamma\big|_{[t_{i-1}, t_i]}$ is a smooth arc $\forall i$.



Def Let $\gamma : [a,b] \to S$ be a contour in a subset $S \subset \mathbb{C}$ and let $f : S \to \mathbb{C}$ be a function. Then the contour integral of f along γ is

$$\int_\gamma f(z)\,dz := \int_a^b (f \circ \gamma)(t)\,\gamma'(t)\,dt$$

In practice, we write $\gamma(t) = x(t) + iy(t)$ and $f(z) = u(x,y) + iv(x,y)$. Then

$$\int_\gamma f(z)\,dz = \int_a^b \big(u(x(t),y(t)) + iv(x(t),y(t)) \big) \cdot \big(x'(t) + iy'(t) \big)\,dt$$

$$= \int_a^b \left[u(x(t),y(t))\,x'(t) - v(x(t),y(t))\,y'(t) \right] dt$$

$$+ i \int_a^b \left[v(x(t),y(t))\,x'(t) + u(x(t),y(t))\,y'(t) \right] dt$$

$$= \int_\gamma (u\,dx - v\,dy) + i \int_\gamma (v\,dx + u\,dy)$$

<u>Rmk</u> The integral is independent of the choice of (orientation-preserving) parametrization of the contour. (Why?)

e.g. • $f(z) = \frac{1}{z - z_0}$ along $\gamma : [0, 2\pi] \to \mathbb{C}$ defined by $\gamma(t) = z_0 + re^{it}$

(i.e. the circle $\{z \in \mathbb{C} : |z - z_0| = r\}$)

$$\int_\gamma f(z)\,dz = \int_0^{2\pi} \frac{1}{r} e^{-it} \cdot ire^{it}\,dt = \int_0^{2\pi} i\,dt = 2\pi i.$$

• $f(z) = \frac{1}{(z - z_0)^n}$ where $n \geq 2$ with γ as above.

$$\int_\gamma f(z)\,dz = \int_0^{2\pi} \frac{1}{r^{n-1}}\, ie^{-i(n-1)t}\,dt = -\frac{1}{r^{n-1}(n-1)}\left[e^{-i(n-1)t}\right]_0^{2\pi} = 0$$

Prop Suppose $\exists M>0$ s.t. $|f(\gamma(t))| \leq M \quad \forall t \in [a,b]$. Then

$$\left| \int_\gamma f(z)\,dz \right| \leq ML$$

where $L := \int_a^b |\gamma'(t)|\,dt$ is the length of the contour γ

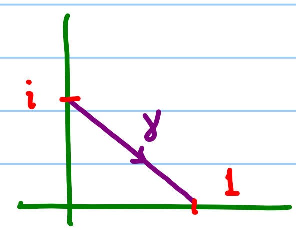
Pf :

$$\left| \int_\gamma f(z)\,dz \right| = \left| \int_a^b (f\circ\gamma)(t)\,\gamma'(t)\,dt \right|$$

$$\leq \int_a^b |f(\gamma(t))|\,|\gamma'(t)|\,dt$$

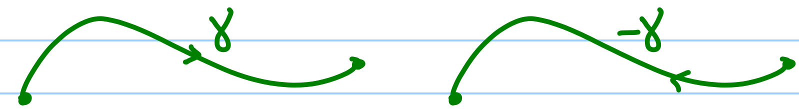
$$\leq \int_a^b M\,|\gamma'(t)|\,dt = ML. \ \#$$

e.g. $f(z) = \frac{1}{z^4}$ and γ is the line segment from i to 1.

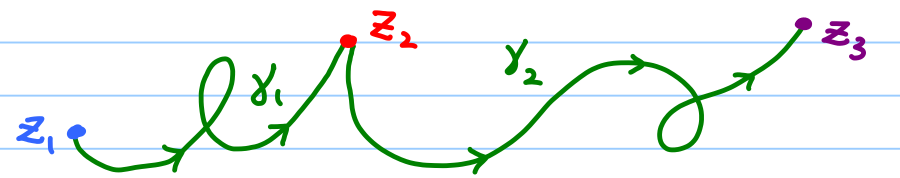


Since $|z| \geq \frac{1}{\sqrt{2}}$ on γ, $|f(z)| \leq 4 \implies \left| \int_\gamma f(z)\,dz \right| \leq 4\sqrt{2}$

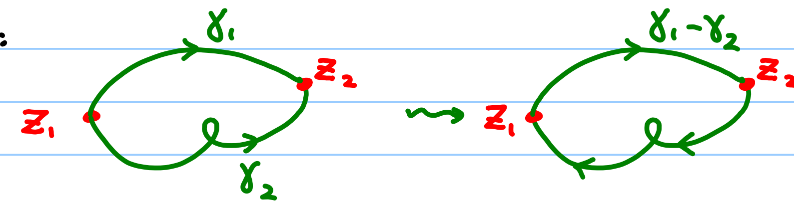
- Let $\gamma : [a,b] \to \mathbb{C}$ be a contour. Then $-\gamma : [-b,-a] \to \mathbb{C}$ denotes the contour defined by $-\gamma(t) := \gamma(-t)$



- If γ_1 is a contour from z_1 to z_2 and γ_2 is a contour from z_2 to z_3, then $\gamma_1 + \gamma_2$ is the contour from z_1 to z_3 given by concatenating γ_1 and γ_2 at z_2:



- If γ_1 is a contour from z_1 to z_2 and γ_2 is a contour from z_3 to z_2, then $\gamma_1 - \gamma_2$ is defined as $\gamma_1 + (-\gamma_2)$:



Prop (1) $\int_\gamma (A f(z) + B g(z))\, dz = A \int_\gamma f(z)\, dz + B \int_\gamma g(z)\, dz$ for consts A, B

(2) $\int_{-\gamma} f(z)\, dz = -\int_\gamma f(z)\, dz$

(3) $\int_{\gamma_1+\gamma_2} f(z)\, dz = \int_{\gamma_1} f(z)\, dz + \int_{\gamma_2} f(z)\, dz$

Def Let $f(z)$ be a $\mathbb{C}$-valued function on a domain $D \subset \mathbb{C}$.
An antiderivative of $f(z)$ on D is a function $F(z)$
s.t. $F'(z) = f(z)$ $\forall z \in D$.

Rmk
- An antiderivative is automatically analytic
- If $F_1' = F_2' = f$, then $F_1 = F_2 + \text{const}$

Thm Let $f(z)$ be a continuous $\mathbb{C}$-valued function on a domain $D \subset \mathbb{C}$.
Then the following are equivalent (abbrev. TFAE):

(a) $f(z)$ has an antiderivative $F(z)$ on D.

(b) For any contours γ_1, γ_2 in D with the same initial and end points,

$$\int_{\gamma_1} f(z)\,dz = \int_{\gamma_2} f(z)\,dz \qquad \text{(independent of path)}$$

(c) For any closed contour γ in D,

$$\int_{\gamma} f(z)\,dz = 0$$

Pf: (b) $\Rightarrow$ (c): Take γ' as a const contour at the initial (= end) pt of γ. Then

$$\int_{\gamma} f(z)\,dz = \int_{\gamma'} f(z)\,dz = 0$$

(c) $\Rightarrow$ (b) : Take $\gamma := \gamma_1 - \gamma_2$. Then γ is closed, so

$$\int_{\gamma_1} f(z)\,dz - \int_{\gamma_2} f(z)\,dz = \int_\gamma f(z)\,dz = 0$$

(a) $\Rightarrow$ (b) : Let z_1, z_2 be any pts in D and $\gamma \subset D$ any contour from z_1 to z_2.

Then $\int_\gamma f(z)\,dz = \int_\gamma F'(z)\,dz = F(z_2) - F(z_1)$

(b) $\Rightarrow$ (a) (most difficult step): Fix $z_0 \in D$. Then for any $z \in D$, we set

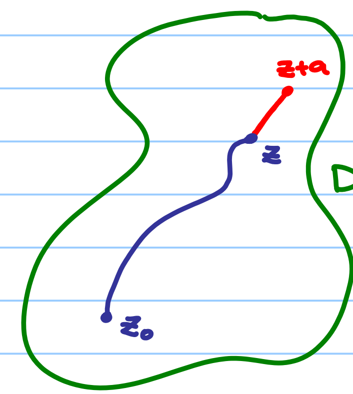
$$F(z) := \int_\gamma f(z)\,dz = \int_{z_0}^{z} f(z)\,dz$$

where γ is any contour from z_0 to z in D.

By (b), F is well-defined.

Now $F(z+a) - F(z) = \int_{z_0}^{z+a} f(w)\,dw - \int_{z_0}^{z} f(w)\,dw = \int_{z}^{z+a} f(w)\,dw$

$$\Rightarrow \frac{F(z+a)-F(z)}{a} - f(z) = \int_z^{z+a} \frac{f(w)-f(z)}{a}\,dw$$



Since f is continuous,

$\forall \varepsilon > 0,\ \exists \delta > 0$ s.t. $|f(w) - f(z)| < \varepsilon$ for $|w-z| < \delta$

When $|a|$ is small enough, we can use a line segment to connect z to $z+a$.

So then

$$\left| \frac{F(z+a)-F(z)}{a} - f(z) \right| = \left| \int_z^{z+a} \frac{f(w)-f(z)}{a}\,dw \right| \leq \frac{\varepsilon}{|a|} \cdot |a| = \varepsilon.$$

Hence $F'(z) = f(z)$. #