MMAT 5220 Complex Analysis and Its Applications

Lecture 2

§ Functions on $\mathbb{C}$

We will study complex-valued functions defined on subsets in $\mathbb{C}$:

$$f : \underset{\mathbb{C}}{\underset{\cap}{S}} \longrightarrow \mathbb{C}$$

- S is called the domain of definition of f
- $f(S) := \{ f(z) : z \in S \} \subset \mathbb{C}$ is called the image of f
- We often write $f = u + iv$ where $u = \mathrm{Re} f$ and $v = \mathrm{Im} f$.

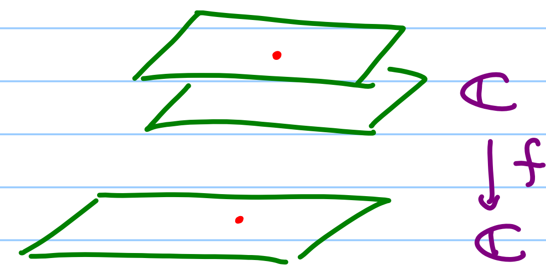
  Then $\begin{cases} u = u(x,y) \\ v = v(x,y) \end{cases}$ are real-valued functions for $z = x + iy$.

e.g. Consider $f : \mathbb{C} \to \mathbb{C}$ defined by $w = f(z) = z^2$.

Writing $u + iv = (x+iy)^2 = (x^2 - y^2) + i\,2xy$.

So f can be described by

$$\begin{cases} u = x^2 - y^2 \\ v = 2xy \end{cases}$$



How to visualize f? We may solve x, y in terms of u, v:

$$x = \pm\sqrt{\frac{\sqrt{u^2+v^2}+u}{2}}, \quad y = \pm\sqrt{\frac{\sqrt{u^2+v^2}-u}{2}}$$

(the signs depend on that of v)

$\rightsquigarrow$ Preimage of any $u+iv \neq 0$ is 2 pts; while that of 0 is 1 pt

§ Limits and continuity

Consider a function $f : S \to \mathbb{C}$ on a subset $S \subset \mathbb{C}$.

Def Let $z_0 \in S$. We say that f has a limit w_0 as $z \to z_0$, denoted as

$$\lim_{z \to z_0} f(z) = w_0 ,$$

if $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $|f(z) - w_0| < \varepsilon$ when $z \in S$ & $|z - z_0| < \delta$.

Rmk: (optional) Actually it also makes sense to talk about limit of f as z approaches a pt at the boundary ∂S of S.

e.g. $\lim_{z \to 1} \operatorname{Log} z = 0$ but $\lim_{z \to 0} \operatorname{Log} z$ doesn't exist

Prop If $\lim_{z \to z_0} f(z)$ exists, then it is unique

Pf: Suppose $w_0 = \lim_{z \to z_0} f(z) = w_1$. Then $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. if $|z - z_0| < \delta$

then $|f(z) - w_0| < \varepsilon$ & $|f(z) - w_1| < \varepsilon$

$\Rightarrow |w_0 - w_1| \leq |f(z) - w_0| + |f(z) - w_1| < 2\varepsilon$ by the $\triangle$ ineq.

Letting $\varepsilon \to 0$, we have $w_0 = w_1$. #

Prop Let $f(z) = u(x,y) + i v(x,y)$ where $z = x + iy$.

Then $\lim_{z \to z_0} f(z) = w_0$ iff

$$\lim_{(x,y) \to (x_0,y_0)} u(x,y) = u_0 \quad \text{and} \quad \lim_{(x,y) \to (x_0,y_0)} v(x,y) = v_0$$

where we write $z_0 = x_0 + iy_0$, $w_0 = u_0 + iv_0$.

Pf: ($\Leftarrow$): Let $\varepsilon > 0$. Then $\exists\, \delta > 0$ s.t. when $\mathrm{dist}((x,y),(x_0,y_0)) = |z - z_0| < \delta$

we have $|u(x,y) - u_0| < \frac{\varepsilon}{2}$ & $|v(x,y) - v_0| < \frac{\varepsilon}{2}$

$\Rightarrow |f(z) - w_0| \leq |u(x,y) - u_0| + |v(x,y) - v_0| < \varepsilon$ by the Δ ineq.

So $\lim_{z \to z_0} f(z) = w_0$.

($\Rightarrow$): Exercise. #

Prop Suppose that $\lim_{z \to z_0} f(z) = w_0$ and $\lim_{z \to z_0} g(z) = t_0$. Then

(1) $\lim_{z \to z_0} (f(z) \pm g(z)) = w_0 \pm t_0$.

(2) $\lim_{z \to z_0} f(z) g(z) = w_0 t_0$.

(3) $\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{w_0}{t_0}$ if $t_0 \neq 0$.

Def A function $f : S \longrightarrow \mathbb{C}$ is said to be continuous at $z_0 \in S$ if $\lim_{z \to z_0} f(z) = f(z_0)$ (meaning $\lim_{z \to z_0} f(z)$ exists, $f(z_0)$ is well-defined and the values are equal)

If f is continuous at every pt in S, we say f is a continuous function on S.

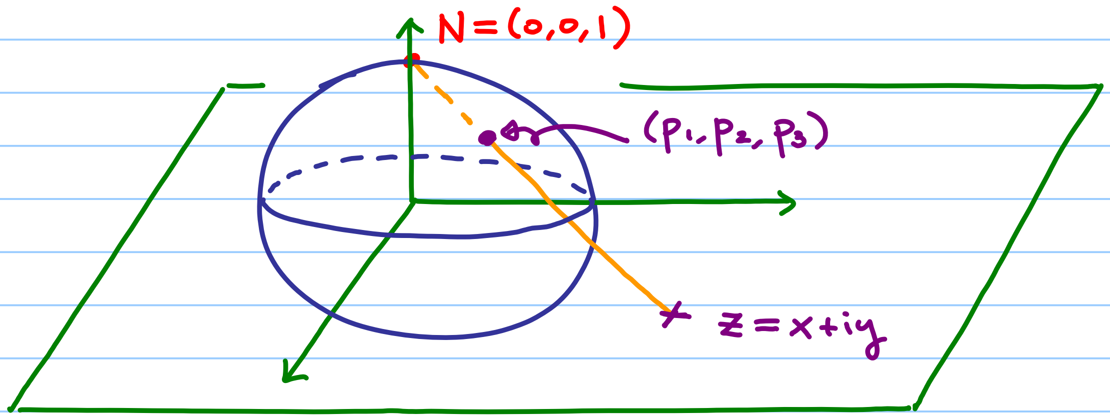
Cor • Let $f(z) = u(x,y) + i\,v(x,y)$ where $z = x + iy$.
Then f is continuous at $z_0 = x_0 + iy_0$ iff u and v are continuous at (x_0, y_0).

• If $f, g : S \longrightarrow \mathbb{C}$ are continuous at $z_0 \in S$, then $f \pm g$, fg and $\frac{f}{g}$ (if $g(z_0) \neq 0$) are continuous at z_0.

e.g. All the elementary functions are continuous on their domains of definition.

$\boxed{\text{Limits involving } \infty}$

First we have the stereographic projection:



This defines a map

$$\varphi : \mathbb{C} \longrightarrow S^2 \leftarrow \text{unit sphere in } \mathbb{R}^3$$

Explicitly, for $z = x+iy$

$$(p_1, p_2, p_3) = \left(\frac{2x}{|z|^2+1}, \frac{2y}{|z|^2+1}, \frac{|z|^2-1}{|z|^2+1} \right)$$

with inverse $\varphi^{-1} : S^2 \setminus \{N\} \longrightarrow \mathbb{C}$

$$z = x+iy = \frac{p_1 + i p_2}{1 - p_3}$$

$\rightsquigarrow$ The extended complex plane $\mathbb{CP}^1 := \mathbb{C} \cup \{\infty\} \cong \mathbb{S}^2$

Def $\lim\limits_{z\to z_0} f(z) = \infty$ if $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $|f(z)| > \frac{1}{\varepsilon}$ when $|z - z_0| < \delta$.

So $\lim\limits_{z\to z_0} f(z) = \infty$ iff $\lim\limits_{z\to z_0} \frac{1}{f(z)} = 0$

Def $\lim\limits_{z\to\infty} f(z) = w_0$ if $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $|f(z) - w_0| < \varepsilon$ when $|z| > \frac{1}{\delta}$

(or $|f(\frac{1}{w}) - w_0| < \varepsilon$ when $|w| < \delta$)

So $\lim\limits_{z\to\infty} f(z) = w_0$ iff $\lim\limits_{w\to 0} f(\frac{1}{w}) = w_0$

Combined together, we have

$$\lim_{z\to\infty} f(z) = \infty \iff \forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } |f(z)| > \frac{1}{\varepsilon} \text{ when } |z| > \frac{1}{\delta}$$

$$\iff \lim_{w\to 0} \frac{1}{f(\frac{1}{w})} = 0$$

Some terminologies

- $B(z_0,\rho) := \{z \in \mathbb{C} : |z - z_0| < \rho\}$ is called an (open) disk centered at $z_0 \in \mathbb{C}$ with radius $\rho > 0$
- A subset $U \subset \mathbb{C}$ is open if $\forall z_0 \in U$, $\exists \rho > 0$ s.t. $B(z_0,\rho) \subset U$
- U is called an (open) neighborhood of $z_0 \in \mathbb{C}$ if U is open and $z_0 \in U$.
- A subset $T \subset \mathbb{C}$ is closed if $\mathbb{C} \setminus T$ is open.
- A subset $D \subset \mathbb{C}$ is a domain if D is open and connected

  (i.e., $\forall z_0, z_1 \in D$, $\exists$ a continuous path $\gamma : [0,1] \to D$ s.t. $\gamma(0) = z_0$, $\gamma(1) = z_1$).

§ Differentiability

Let $f: S \to \mathbb{C}$ be a function where S contains a nbh of a pt $z_0 \in S$.

Def We say f is (complex) differentiable at z_0 if the limit

$$f'(z_0) := \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \quad \text{exists;}$$

in this case, $f'(z_0)$ is called the derivative of f at z_0.

e.g. Let $f(z) = z^n$, $n \in \mathbb{N}$. Then

$$\lim_{a \to 0} \frac{f(z+a) - f(z)}{a} = \lim_{a \to 0} \frac{(z+a)^n - z^n}{a} = \lim_{a \to 0} \left(nz^{n-1} + \binom{n}{2} a z^{n-2} + \cdots \right) = nz^{n-1}$$

So $f'(z) = nz^{n-1}$

e.g. Let $f(z) = \frac{1}{z}$. Then

$$\lim_{a\to 0} \frac{f(z+a)-f(z)}{a} = \lim_{a\to 0} \frac{\frac{1}{z+a} - \frac{1}{z}}{a} = \lim_{a\to 0} \frac{-1}{z(z+a)} = -\frac{1}{z^2}$$

e.g. Let $f(z) = \bar{z}$. Then

$$\lim_{a\to 0} \frac{f(z+a)-f(z)}{a} = \lim_{a\to 0} \frac{\bar{a}}{a} = \lim_{a\to 0} e^{-2i\arg a}$$

which doesn't exist.

e.g. Let $f(z) = |z|^2$. Then

$$\lim_{a\to 0} \frac{f(z+a)-f(z)}{a} = \lim_{a\to 0} \frac{|z+a|^2 - |z|^2}{a} = \lim_{a\to 0} \left(\bar{z} + \bar{a} + z\frac{\bar{a}}{a}\right)$$

which exists only when $z = 0$.

Rmk Writing $|z|^2 = f(z) = u(x,y) + iv(x,y)$, we have

$$u(x,y) = x^2 + y^2 \quad \text{and} \quad v(x,y) = 0,$$

both of which are infinitely differentiable. In particular, differentiability of u and v DO NOT imply differentiability of f.

Prop If f is differentiable at z_0, then f is continuous at z_0.

Pf:

$$\lim_{z \to z_0} (f(z) - f(z_0)) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \cdot \lim_{z \to z_0} (z - z_0) = f'(z_0) \cdot 0 = 0$$

$$\Rightarrow \lim_{z \to z_0} f(z) = f(z_0). \quad \#$$

Rmk Converse of the above proposition is not true.

Prop 1) $(f \pm g)'(z) = f'(z) \pm g'(z)$

2) $(fg)'(z) = f'(z)g(z) + f(z)g'(z)$

3) $\left(\frac{f}{g}\right)'(z) = \frac{f'(z)g(z) - f(z)g'(z)}{g(z)^2}$

4) (Chain rule) If g is differentiable at z_0 and f is differentiable at $g(z_0)$, then $f \circ g$ is differentiable at z_0 and

$$(f \circ g)'(z_0) = f'(g(z_0)) \cdot g'(z_0).$$

Idea of Pf : 2) :

$$\frac{f(z+a)g(z+a) - f(z)g(z)}{a} = f(z+a) \cdot \frac{g(z+a) - g(z)}{a} + \frac{f(z+a) - f(z)}{a} \cdot g(z)$$

4) :

$$\frac{f(g(z)) - f(g(z_0))}{z - z_0} = \frac{f(g(z)) - f(g(z_0))}{g(z) - g(z_0)} \cdot \frac{g(z) - g(z_0)}{z - z_0}. \quad \#$$

§ Cauchy Riemann equations

Thm Let $f(z) = u(x,y) + iv(x,y)$. Then $f'(z_0)$ exists at a pt $z_0 = x_0 + iy_0$ if and only if the map $F: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $(x,y) \longmapsto (u(x,y), v(x,y))$ is differentiable at (x_0,y_0) and satisfies the Cauchy-Riemann eqns:

$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases} \quad \text{at } (x_0,y_0);$$

in this case, $f'(z_0) = u_x(x_0,y_0) + iv_x(x_0,y_0)$.

Rmk Recall that a map $F: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is differentiable at a pt $\vec{p_0} = (x_0,y_0)$ iff $\exists$ a linear map (the Jacobian matrix) $J: \mathbb{R}^2 \to \mathbb{R}^2$

s.t. $\displaystyle\lim_{\|\vec{a}\| \to 0} \frac{\| F(\vec{p_0} + \vec{a}) - F(\vec{p_0}) - J(\vec{a}) \|}{\|\vec{a}\|} = 0.$ (Here $\|\cdot\|$ is the norm in $\mathbb{R}^2$)

Pf: $f'(z_0)$ exists $\iff \lim_{a\to 0} \dfrac{f(z_0+a)-f(z_0)}{a} = f'(z_0)$

$$\iff \lim_{a\to 0} \left| \frac{f(z_0+a)-f(z_0)-f'(z_0)a}{a} \right| = 0$$

Writing $a = b+ic$ and $f'(z_0) = \beta + i\gamma$, we have

$$f'(z_0)a = (\beta+i\gamma)(b+ic) = (\beta b - \gamma c) + i(\gamma b + \beta c)$$

So $f'(z)$ exists at $z_0 = x_0 + iy_0$

$$\iff \lim_{|a|\to 0} \frac{\left|[u(x_0+b, y_0+c) - u(x_0,y_0) - (\beta b - \gamma c)] + i[v(x_0+b, y_0+c) - v(x_0,y_0) - (\gamma b + \beta c)]\right|}{|a|} = 0$$

$$\iff \lim_{\|\vec{a}\|\to 0} \frac{1}{\|\vec{a}\|} \left| F(\vec{p}_0+\vec{a}) - F(\vec{p}_0) - \begin{pmatrix} \beta & -\gamma \\ \gamma & \beta \end{pmatrix} \vec{a} \right| = 0$$

(where we write $\vec{a} = \begin{pmatrix} b \\ c \end{pmatrix}$, $\vec{p}_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in \mathbb{R}^2$)

$\Leftrightarrow$ F is differentiable at $\vec{p}_0 = (x_0, y_0)$ w/ Jacobian matrix

$$J = \begin{pmatrix} \beta & -\gamma \\ \gamma & \beta \end{pmatrix}$$

$\Leftrightarrow$ F is differentiable at $\vec{p}_0 = (x_0, y_0)$ s.t. $\begin{cases} u_x = v_y = \beta \\ v_x = -u_y = \gamma \end{cases}$ at (x_0, y_0).

#

e.g. Let

$$f(z) = \begin{cases} \dfrac{\bar{z}^2}{z} & \text{for } z \neq 0 \\ 0 & \text{for } z = 0 \end{cases}.$$

Then for $z \neq 0$, $f(z) = \dfrac{x^3 - 3xy^2}{x^2 + y^2} + i\,\dfrac{-3x^2y + y^3}{x^2 + y^2}$

i.e. $u(x,y) = \begin{cases} \dfrac{x^3 - 3xy^2}{x^2+y^2} & \text{for } (x,y) \neq (0,0) \\ 0 & \text{for } (x,y) = (0,0) \end{cases}$, $v(x,y) = \begin{cases} \dfrac{-3x^2y + y^3}{x^2+y^2} & \text{for } (x,y) \neq (0,0) \\ 0 & \text{for } (x,y) = (0,0) \end{cases}$

One can check that $u_x(0,0) = 1 = v_y(0,0)$, $u_y(0,0) = 0 = -v_x(0,0)$, so the CR eqns are satisfied.

But $\dfrac{f(a)-f(0)}{a} = \left(\dfrac{\bar{a}}{a}\right)^2$ whose limit as $a \to 0$ doesn't exist and hence f is not cpx differentiable at 0.

<u>Cor</u> Suppose $f(z) = u(x,y) + iv(x,y)$ is defined in some open nbd U (e.g. $B(z_0,\rho)$) of $z_0 = x_0 + iy_0 \in \mathbb{C}$, and u_x, u_y, v_x, v_y exist in that nbd and are continuous at (x_0,y_0). If u, v satisfy the CR eqns at (x_0,y_0), then $f'(z_0)$ exists.

Pf : The conditions that u_x, u_y, v_x, v_y exist in the nbd and are continuous at (x_0, y_0) imply that $F(x,y) = \begin{pmatrix} u \\ v \end{pmatrix}$ is differentiable at $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$

So the result follows from the previous Thm. #