MMAT 5220 Complex Analysis and Its Applications

Lecture 12

§ Conformal mappings (cont'd)

Def A function f, with continuous partial derivatives, is said to be **conformal at z_0** if, for any paths $\gamma_1, \gamma_2 : (-\varepsilon, \varepsilon) \to \mathbb{C}$ with $\gamma_1(0) = \gamma_2(0) = z_0$, the (oriented) angle between γ_1 and γ_2 at z_0 is equal to that between $f \circ \gamma_1$ and $f \circ \gamma_2$ at $f(z_0)$.

A function is said to be **conformal in a domain $D \subset \mathbb{C}$** if it is conformal at every point in D

Thm A function f is conformal at z_0 iff f is complex differentiable at z_0 and $f'(z_0) \neq 0$.

Pf : ($\Leftarrow$) $\arg(f\circ\gamma_2)'(0) - \arg(f\circ\gamma_1)'(0) = (\arg f'(z_0) + \arg\gamma_2'(0)) - (\arg f'(z_0) + \arg\gamma_1'(0))$

$$= \arg\gamma_2'(0) - \arg\gamma_1'(0).$$

($\Rightarrow$) Note that f is conformal at z_0 iff

$$\arg(f\circ\gamma)'(0) - \arg\gamma'(0) = \arg\frac{(f\circ\gamma)'(0)}{\gamma'(0)}$$

is independent the choice of $\gamma : (-\varepsilon, \varepsilon) \to \mathbb{C}$ with $\gamma(0) = z_0$.

Writing $\gamma(t) = x(t) + iy(t)$, we have

$$(f\circ\gamma)'(0) = \frac{\partial f}{\partial x}x'(0) + \frac{\partial f}{\partial y}y'(0)$$

$$= \frac{\partial f}{\partial x}\left(\frac{\gamma'(0) + \bar{\gamma}'(0)}{2}\right) - i\frac{\partial f}{\partial y}\left(\frac{\gamma'(0) - \bar{\gamma}'(0)}{2}\right)$$

$$= \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)\gamma'(0) + \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)\bar{\gamma}'(0)$$

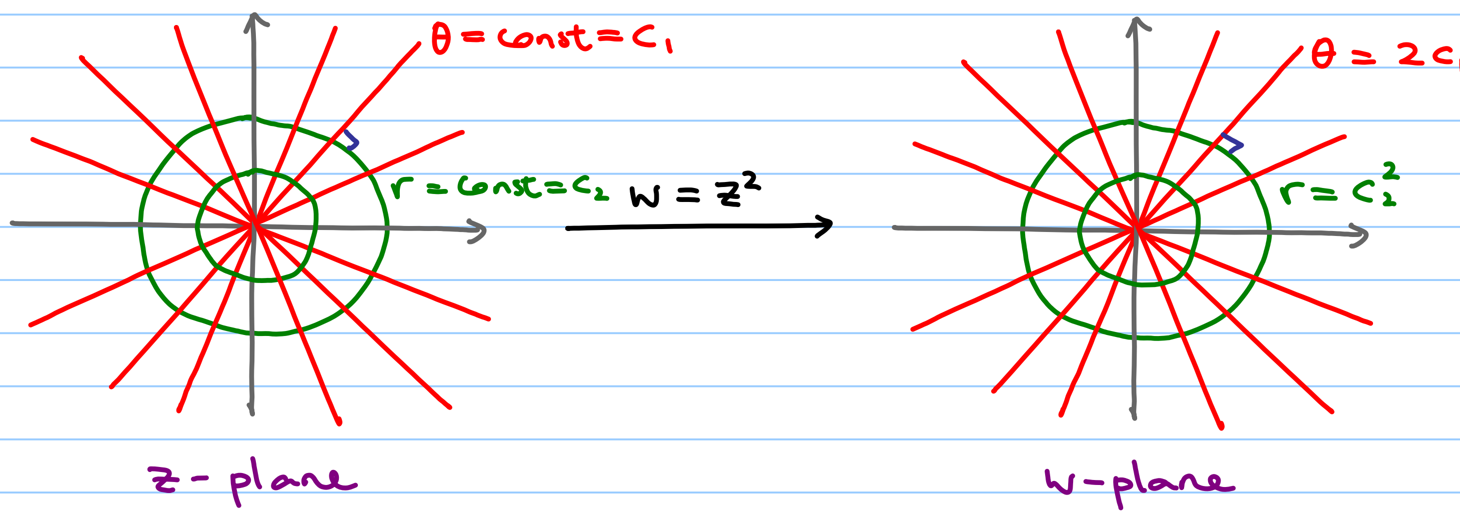
$$\Rightarrow \quad \frac{(f\circ\gamma)'(0)}{\gamma'(0)} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) + \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)\frac{\overline{\gamma'(0)}}{\gamma'(0)}$$

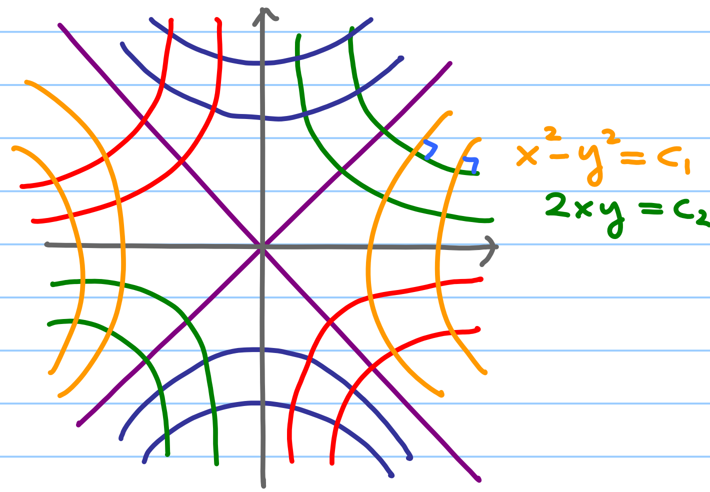
As $\gamma'(0)$ varies, the RHS describes a circle with radius $\frac{1}{2}\left|\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right|$

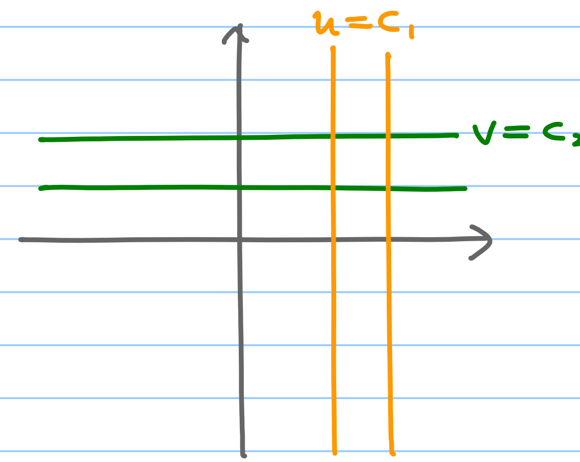
So we must have $\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y} = 0 \iff$ CR $eq^{\underline{ns}}$ at z_0. #

(why?)

e.g. $f(z) = z^2$







level curves $u=$ const and $v=$ const are orthogonal everywhere except at $z=0$

(Note: $f'(0)=0$!)

$$\begin{cases} u(x,y) = x^2-y^2 \\ v(x,y) = 2xy \end{cases}$$

- Let $\mathbb{D}$ be the open unit disk $\{z \in \mathbb{C} : |z| < 1\}$.
Consider $g : \mathbb{D} \to \mathbb{C}$ defined by

$$g(z) = \frac{z-a}{1-\bar{a}z}$$

where $a \in \mathbb{D}$. Then one can check that

1. $g(\mathbb{D}) \subset \mathbb{D}$
2. $g'(z) \neq 0 \quad \forall z \in \mathbb{D}$
3. $g : \mathbb{D} \to \mathbb{D}$ is a bijection

Def A fractional linear transformation is a function of the form

$$f(z) = \frac{az+b}{cz+d}$$

where $a,b,c,d \in \mathbb{C}$ are s.t. $ad-bc \neq 0$.

Properties:

- f is conformal
- f maps circles/straight lines to circles/straight lines in $\mathbb{C}$ (Viewing f as a map from the Riemann sphere $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$ to itself, then f maps circles to circles in $\mathbb{CP}^1$).
- f is a composition of $\left\{ \begin{array}{ll} \text{- dilations} & z \mapsto az \\ \text{- translations} & z \mapsto a+z \\ \text{- inversions} & z \mapsto 1/z \end{array} \right.$ $\Rightarrow$ f is bijective

To view f as a map from the Riemann sphere $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$ to itself, we set

$$\begin{cases} f(-d/c) = \infty & \text{if } c \neq 0 \\ f(\infty) = \infty & \text{if } c = 0 \end{cases}$$

Then $f : \mathbb{CP}^1 \longrightarrow \mathbb{CP}^1$ is actually a biholomorphism, i.e. f is invertible and its inverse is also analytic.

Furthermore, any biholomorphism of $\mathbb{CP}^1$ is of this form, i.e.

$$\mathrm{Aut}(\mathbb{CP}^1) = \left\{ f(z) = \frac{az+b}{cz+d} : a, b, c, d \in \mathbb{C} \text{ and } ad-bc \neq 0 \right\}$$

So we have a map

$$\pi : GL(2,\mathbb{C}) \longrightarrow \mathrm{Aut}(\mathbb{CP}^1)$$
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto f(z) = \frac{az+b}{cz+d}$$

which is a surjective group homomorphism whose kernel is given by

$$\operatorname{Ker} \pi = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} : \lambda \in \mathbb{C} \setminus \{0\} \right\}$$

$$\Rightarrow \operatorname{Aut}(\mathbb{CP}^1) \cong GL(2, \mathbb{C}) / (\mathbb{C} \setminus \{0\}) =: PGL(2, \mathbb{C}).$$

§ Schwarz Lemma

Thm (Schwarz Lemma)

Let $f(z)$ be an analytic function for $|z|<1$ such that $f(0)=0$ and $|f(z)|\leq 1$ $\forall\ |z|<1$. Then

$$\begin{cases} |f(z)| \leq |z| \quad \forall\, |z|<1 \quad \text{and} \\ |f'(0)| \leq 1 \end{cases}$$

Furthermore, if $|f'(0)|=1$ or $|f(z_0)|=|z_0|$ for some z_0 with $0<|z_0|<1$, then $f(z)=e^{i\theta}z$ for some $\theta\in\mathbb{R}$.

Pf: Define $g(z)$ by $g(z) := \begin{cases} \dfrac{f(z)}{z} & \text{for } 0<|z|<1 \\ f'(0) & \text{for } z=0 \end{cases}$

Then g is analytic for $|z|<1$. (Why?)

For any $r<1$, applying maximum modulus principle to $g(z)$ for $|z|\le r$ gives

$$|g(z)| \le \max_{|z|=r} |g(z)| = \max_{|z|=r} \frac{|f(z)|}{|z|} \le \frac{1}{r}.$$

Letting $r\to 1$, we have $|g(z)|\le 1$

$$\Rightarrow \begin{cases} |f(z)| \le |z| \quad \forall |z|<1 \text{ and} \\ |f'(0)| \le 1. \end{cases}$$

If $|f'(0)|=1$ or $\exists\, z_0$ with $0<|z_0|<1$ s.t. $|f(z_0)|=|z_0|$, then either $|g(0)|=1$ or $|g(z_0)|=1$. In both cases, the maximum modulus principle implies that $g(z)\equiv\lambda$ where $|\lambda|=1$.

The result follows. #

Thm Let $\mathbb{D}$ be the open unit disk $\{z \in \mathbb{C} : |z| < 1\}$.
Then $g : \mathbb{D} \to \mathbb{D}$ is conformal bijective iff g is of the form

$$g(z) = e^{i\theta_0} \frac{z-a}{1-\bar{a}z} \quad \text{for some } a \in \mathbb{C} \text{ with } |a|<1 \text{ and } \theta_0 \in \mathbb{R}$$

Pf : ($\Leftarrow$) By direct computations (see Lecture 11).

($\Rightarrow$) Suppose $g : \mathbb{D} \to \mathbb{D}$ is conformal bijective.

Let $a = g^{-1}(0) \in \mathbb{C}$ or $g(a) = 0$.

Consider the map $h : \mathbb{D} \to \mathbb{D}$ defined by $h(z) = \frac{z-a}{1-\bar{a}z}$.

Then $f := g \circ h^{-1} : \mathbb{D} \to \mathbb{D}$ is conformal bijective s.t. $f(0) = 0$.

Applying Schwarz Lemma to f (resp. f^{-1}) gives

$$|f(z)| \leq |z| \quad (\text{resp. } |f^{-1}(w)| \leq |w| \Rightarrow |z| \leq |f(z)|)$$

Hence we have $|f(z)| = |z| \Rightarrow f(z) = e^{i\theta_0} z$ for some $\theta_0 \in \mathbb{R}$.

So $g(z) = (f \circ h)(z) = e^{i\theta_0} \dfrac{z-a}{1-\bar{a}z}$. #

§ Riemann Mapping Theorem

Thm (Riemann Mapping Theorem)

If D is a simply-connected domain in $\mathbb{C}$ s.t. $D \neq \mathbb{C}$, then there exists a bijective conformal map $\varphi : D \longrightarrow \mathbb{D}$

Rmks • Suppose $\varphi_1, \varphi_2 : D \longrightarrow \mathbb{D}$ are conformal and bijective.

Then $\varphi_2 \circ \varphi_1^{-1} : \mathbb{D} \longrightarrow \mathbb{D}$ is also bijective conformal.

$$\Longrightarrow \quad (\varphi_2 \circ \varphi_1^{-1})(z) = e^{i\theta_0} \frac{z-a}{1-\bar{a}z} =: g(z)$$

$$\Longrightarrow \quad \varphi_2 = g \circ \varphi_1$$

So φ_1 determines all other $\varphi : D \longrightarrow \mathbb{D}$.

$$\begin{tikzcd} & D \arrow[dl, "\varphi_1"'] \arrow[dr, "\varphi_2"] & \\ \mathbb{D} \arrow[rr, "\varphi_2 \circ \varphi_1^{-1}"] & & \mathbb{D} \end{tikzcd}$$

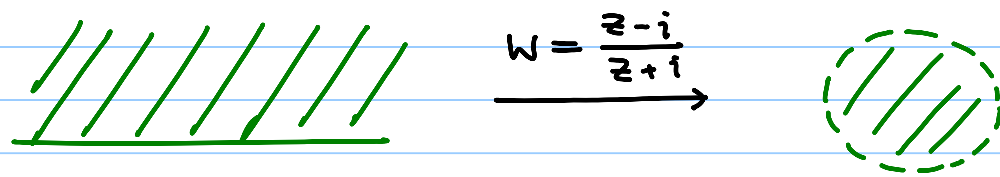
- Similarly, given $\varphi : D \to \mathbb{D}$ bijective conformal.
  Then for any automorphism $f : D \to D$, there exists $g \in \mathrm{Aut}(\mathbb{D})$ determined by the following diagram

$$\begin{array}{ccc} D & \xrightarrow{f} & D \\ \varphi \downarrow & & \downarrow \varphi \\ \mathbb{D} & \xrightarrow{g} & \mathbb{D} \end{array}$$

  i.e. $g = \varphi \circ f \circ \varphi^{-1}$ or $f = \varphi^{-1} \circ g \circ \varphi$.
  So $\mathrm{Aut}(D) \cong \mathrm{Aut}(\mathbb{D})$.

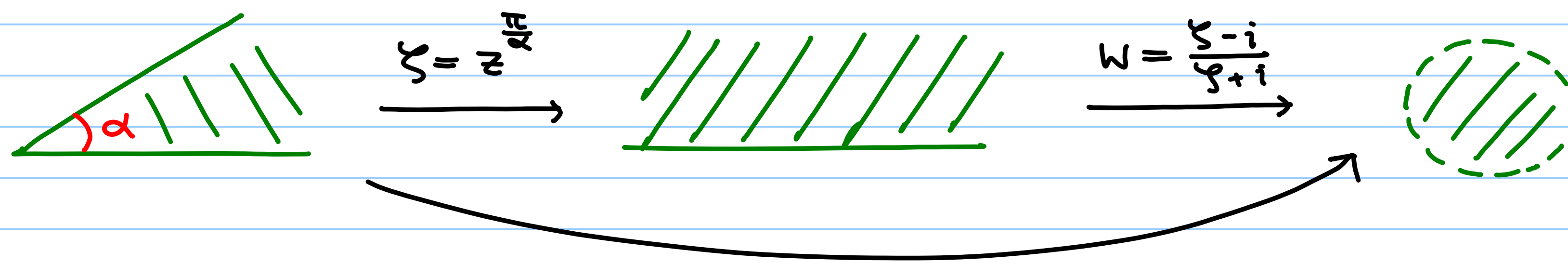
Examples

- Upper half plane $\mathbb{H} = \{ z \in \mathbb{C} : \operatorname{Im} z > 0 \}$



$$w = \frac{z-i}{z+i}$$

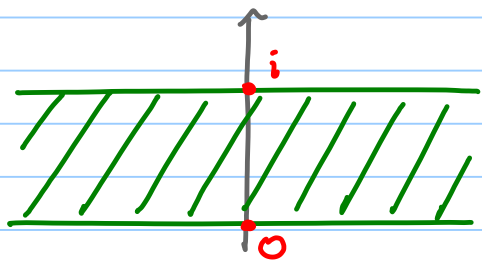
- Sector



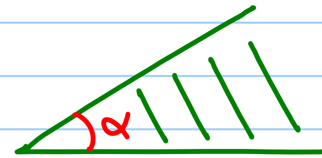
$$\zeta = z^{\frac{\pi}{\alpha}} \qquad w = \frac{\zeta - i}{\zeta + i}$$

$$w = \frac{z^{\pi/\alpha} - i}{z^{\pi/\alpha} + i}$$

• Strip

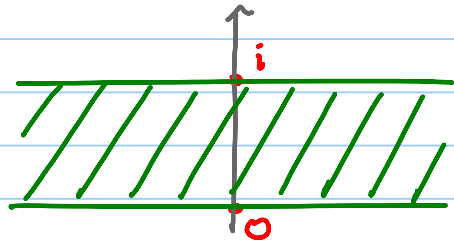


$$w = e^{\alpha z} \longrightarrow$$

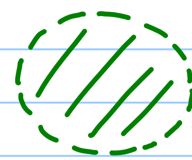


(In particular, choosing $\alpha = \pi$ gives the upper half-plane)

$\Rightarrow$



$$w = \frac{e^{\pi z} - i}{e^{\pi z} + i} \longrightarrow$$



<u>Rmk</u> The Schwarz-Christoffel transformation