THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MMAT5220 Complex Analysis and Its Applications 2019-20
Week 9 Examples

1. Find the poles and residues of the following functions:

(a) $\dfrac{1}{\sin z}$ (b) $\cot z$ (c) $\dfrac{1}{\sin^2 z}$

Solution. Notice that $\cot z = \frac{\cos z}{\sin z}$. For all functions above, the singular points are the zero set of $\sin z$, which is $\pi\mathbb{Z}$.

Moreover, by proposition in Week 9 lecture,

$$\operatorname*{Res}_{z=\pi n} \frac{1}{\sin z} = \frac{1}{\cos \pi n} = (-1)^n,$$
$$\operatorname*{Res}_{z=\pi n} \cot z = \operatorname*{Res}_{z=\pi n} \frac{\cos z}{\sin z} = \frac{\cos(\pi n)}{\cos(\pi n)} = 1.$$

Every zeros of the function $\sin^2 z$ has order 2. In general, if $f(z)$ has a zero of order 2, then we may compute $\operatorname*{Res}_{z=z_0} \dfrac{1}{f(z)}$ in the following way.

$$\begin{aligned} f(z) &= \frac{f^{(2)}(z_0)}{2!}(z-z_0)^2 + \frac{f^{(3)}(z_0)}{3!}(z-z_0)^3 + \cdots \\ &= \frac{f^{(2)}(z_0)}{2}(z-z_0)^2(1-h(z)), \end{aligned}$$

where

$$h(z) = -\sum_{k=1}^{\infty} \frac{2}{f^{(2)}(z_0)} \frac{f^{(k+2)}(z_0)}{(k+2)!}(z-z_0)^k$$

has a zero of order $m \geq 1$ at $z = z_0$. Therefore,

$$\frac{1}{f(z)} = \frac{2}{f^{(2)}(z_0)} \frac{1}{(z-z_0)^2}\left(1 + h(z) + h(z)^2 + \cdots\right)$$

The term $\frac{1}{z-z_0}$ appears only when $h(z)$ is multiplied by $\frac{1}{(z-z_0)^2}$, hence the residue is computed to be

$$\operatorname*{Res}_{z=z_0} \frac{1}{f(z)} = \frac{2}{f^{(2)}(z_0)}\left(-\frac{2}{f^{(2)}(z_0)}\frac{f^{(3)}(z_0)}{3!}\right) = \frac{-2f^{(3)}(z_0)}{3f^{(2)}(z_0)^2}$$

In our case, $f(z) = \sin^2 z$, $f'(z) = \sin 2z$, $f''(z) = 2\cos 2z$, $f^{(3)}(z) = -4\sin 2z$, hence by the formula above $\operatorname*{Res}_{z=\pi n} \dfrac{1}{\sin^2 z} = 0$. ◀

2. Using the residue at infinity to evaluate the integral of $f(z)$ around the positively oriented circle $|z| = 3$ when $f(z)$ equals

(a) $\dfrac{(3z+2)^2}{z(z-1)(2z+5)}$ (b) $\dfrac{z^3(1-3z)}{(1+z)(1+2z^4)}$ (c) $\dfrac{z^3 e^{\frac{1}{z}}}{1+z^3}$

Solution. For each of them, we put $g(z) = \frac{1}{z^2} f(\frac{1}{z})$.

(a) Notice that all poles $z = 0, 1, -5/2$ of $f(z)$ lie inside the contour $|z| = 3$. By Cauchy's residue theorem

$$\int_{|z|=3} f(z)\, dz = 2\pi i \operatorname*{Res}_{z=0} g(z)$$

Note that

$$g(z) = \frac{1}{z^2} \frac{z(3+2z)^2}{(1-z)(2+5z)} = \frac{(3+2z)^2}{z(1-z)(2+5z)}$$

and $\operatorname*{Res}_{z=0} g(z) = 9/2$. Hence $\int_{|z|=3} f(z)\, dz = 9\pi i$.

(b) Note that all poles of the function lie inside the contour $|z| = 3$, and

$$g(z) = \frac{1}{z^2} f(\frac{1}{z}) = \frac{z-3}{z(z+1)(z^4+2)}$$

Hence,

$$\int_{|z|=3} f(z)\, dz = 2\pi i \operatorname*{Res}_{z=0} g(z) = 2\pi i \left(-\frac{3}{2}\right) = -3\pi i$$

(c) Note that the singular points of the function $f(z)$ are $z = 0$ (essential singularity), and $e^{\frac{\pi i}{3}}, e^{\frac{2\pi i}{3}}, 1$ (simple poles), and all of them lie inside the contour $|z| = 3$. Moreover,

$$g(z) = \frac{e^z}{z^2(z^3+1)}$$

To compute $\operatorname*{Res}_{z=0} g(z)$, note that around $z = 0$, we have

$$\frac{e^z}{z^2(z^3+1)} = (\frac{1}{z^2} + \frac{1}{z} + \frac{1}{2} + \frac{1}{3}z + \ldots)(1 - z^3 + z^6 + \cdots)$$

Therefore, we have

$$\int_{|z|=3} f(z)\, dz = 2\pi i \operatorname*{Res}_{z=0} g(z) = 2\pi i.$$

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3. Evaluate the following integrals by the method of residues:

(a) $\int_{-\infty}^{\infty} \frac{x^2-x+2}{x^4+10x^2+9}\, dx$

(b) $\int_0^{\infty} \frac{\cos x}{x^2+a^2}\, dx$, a real,

(c) $\int_0^{\infty} \frac{x \sin x}{x^2+a^2}\, dx$, a real.

Solution.

(a) Consider the positively oriented contour Γ_R composed of the upper semicircle C_R^+ centered at 0 with radius R, and the diameter l_R.
Consider $f(z) = \frac{z^2-z+2}{z^4+10z^2+9}$. By solving $z^4+10z^2+9=0$, the only singular points of $f(z)$ are $z = \pm i, \pm 3i$. For $R > 3$, the only poles lying inside the contour Γ_R are i and $3i$. Using Cauchy's residue theorem, we have

$$\begin{aligned}\int_{\Gamma_R} f(z)\,dz &= 2\pi i(\operatorname*{Res}_{z=i} f(z) + \operatorname*{Res}_{z=3i} f(z))\\ &= 2\pi i\left(\frac{1-i}{2i(4i)(-2i)} + \frac{-3i-7}{4i(2i)(6i)}\right)\\ &= \frac{5\pi}{12}\end{aligned}$$

Now,

$$\begin{aligned}\left|\int_{C_R^+} \frac{z^2-z+2}{z^4+10z^2+9}\,dz\right| &\le \int_{C_R^+} \frac{|z|^2+|z|+2}{|z|^4-10|z|^2-9}dz\\ &= \frac{R^2+R+2}{R^4-10R^2-9}\,\pi R\\ &\to 0 \quad \text{as} \quad R\to\infty\end{aligned}$$

Hence,

$$\int_{-R}^{R} \frac{x^2-x+2}{x^4+10x^2+9}\,dx + \int_{C_R^+} \frac{z^2-z+2}{z^4+10z^2+9}\,dz = \frac{5\pi}{12}$$

implies that

$$\lim_{R\to\infty} \int_{-R}^{R} \frac{x^2-x+2}{x^4+10x^2+9}\,dx = \frac{5\pi}{12}.$$

Indeed, both the improper integrals $\int_0^\infty \frac{x^2-x+2}{x^4+10x^2+9}\,dx$ and $\int_{-\infty}^0 \frac{x^2-x+2}{x^4+10x^2+9}\,dx$ exist, because

$$\left|\frac{x^2-x+2}{x^4+10x^2+9}\right| \le \frac{3x^2}{x^4} = \frac{3}{x^2} \qquad \text{when } |x| \text{ is large enough.}$$

Therefore, the improper integral $\int_{-\infty}^\infty \frac{x^2-x+2}{x^4+10x^2+9}\,dx$ exists and equals $\frac{5\pi}{12}$.

(b) If $a = 0$, then the integral does not exist, because for $0 < x < \frac{\pi}{3}$, $\cos x > \frac{1}{2}$, and hence

$$\int_0^{\frac{\pi}{3}} \frac{\cos x}{x^2} \ge \frac{1}{2}\int_0^{\frac{\pi}{3}} \frac{1}{x^2} = \lim_{x\to 0^+} \frac{1}{2}\left(\frac{1}{x} - \frac{3}{\pi}\right) = \infty.$$

For $a \ne 0$, consider the same contours in part (a) and the function $f(z) = \frac{e^{iz}}{z^2+a^2}$.
Cauchy's residues theorem tells us that for $R > |a|$,

$$\int_{-R}^{R} \frac{e^{ix}}{x^2+a^2} + \int_{C_R^+} \frac{e^{iz}}{z^2+a^2} = 2\pi i \operatorname*{Res}_{z=|a|i} \frac{e^{iz}}{z^2+a^2} = \frac{\pi e^{-|a|}}{|a|}$$

Taking the real part of both sides,

$$\int_{-R}^{R} \frac{\cos x}{x^2+a^2} + \operatorname{Re}\int_{C_R^+} \frac{e^{iz}}{z^2+a^2} = \frac{\pi e^{-|a|}}{|a|}$$

Now, in the upper half plane, we have $y \geq 0$ and hence $|e^{iz}| = e^{-y} \leq 1$. Moreover,

$$\left| \mathrm{Re} \int_{C_R^+} \frac{e^{iz}}{z^2 + a^2} \right| \leq \left| \int_{C_R^+} \frac{e^{iz}}{z^2 + a^2} \right| \leq \frac{\pi R}{R^2 - a^2} \to 0 \quad \text{as } R \to \infty.$$

Since $\frac{\cos x}{x^2+a^2}$ is even, we have

$$\int_0^\infty \frac{\cos x}{x^2 + a^2} = \frac{\pi e^{-|a|}}{2|a|}$$

(c) For $a \neq 0$, consider the integral of $f(z)e^{iz} = \frac{ze^{iz}}{z^2+a^2}$ on Γ_R as in part (a). $|a|i$ is the only singular point inside Γ_R and $\operatorname*{Res}_{z=|a|i} f(z)e^{iz} = \dfrac{e^{-|a|}}{2}$. So Cauchy's residue theorem implies that

$$\int_{-R}^R \frac{xe^{ix}}{x^2 + a^2}\, dx + \int_{C_R^+} \frac{ze^{iz}}{z^2 + a^2}\, dz = \pi i e^{-|a|}$$

On C_R^+, we have

$$\left| \frac{z}{z^2 + a^2} \right| \leq \frac{R}{R^2 - a^2} \to 0 \quad \text{as } R \to \infty$$

By Jordan's Lemma, we have

$$\int_{C_R^+} \frac{ze^{iz}}{z^2 + a^2}\, dz \to 0 \quad \text{as } R \to \infty$$

Taking the imaginary parts, we have

$$\int_0^\infty \frac{x \sin x}{x^2 + a^2}\, dx = \frac{\pi e^{-|a|}}{2}.$$

For $a = 0$, we may compare $\frac{\sin x}{x}$ with $\frac{x \sin x}{x^2+a^2}$ for small a. Let $\epsilon > 0$. It is easy to see that for some $\delta > 0$, we have

$$\left| \int_0^\delta \frac{x \sin x}{x^2 + a^2}\, dx \right| < \epsilon \quad \text{and} \quad \left| \int_0^\delta \frac{\sin x}{x} \right| < \epsilon,$$

because $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1$. For this fixed δ, since $\frac{1}{x^2} - \frac{1}{x^2+a^2} = \frac{a^2}{x^2(x^2+a^2)} \leq \frac{a^2}{x^4}$, we have

$$\left| \int_\delta^\infty \frac{\sin x}{x} - \frac{x \sin x}{x^2 + a^2}\, dx \right| \leq \int_\delta^\infty \frac{a^2}{x^3}\, dx = \frac{a^2}{2\delta^2} \to 0 \quad \text{as } a \to 0$$

This shows that

$$\left| \int_0^\infty \frac{\sin x}{x}\, dx - \int_0^\infty \frac{x \sin x}{x^2 + a^2}\, dx \right| < 2\epsilon \quad \text{for any small } a.$$

Hence,

$$\int_0^\infty \frac{\sin x}{x}\, dx = \lim_{a\to 0^+} \int_0^\infty \frac{x \sin x}{x^2 + a^2}\, dx = \lim_{a \to 0} \frac{\pi e^{-|a|}}{2} = \frac{\pi}{2}.$$

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4. Let $\xi \in \mathbb{R}$. Show that

$$\int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i x\xi}\, dx = e^{-\pi\xi^2}$$

Solution. For the case $\xi = 0$, recall that $\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}$ and hence

$$\int_{-\infty}^{\infty} e^{-\pi x^2}\, dx = \int_{-\infty}^{\infty} e^{-x^2} \frac{dx}{\sqrt{\pi}} = 1.$$

For $\xi < 0$, consider the function $f(z) = e^{-\pi z^2} e^{-2\pi i z\xi}$. Note that for fixed y,

$$|e^{-\pi(x+iy)^2}| = |e^{-\pi(x^2-y^2)-2\pi ixy}| = e^{-\pi(x^2-y^2)} \to 0 \qquad \text{as } x \to \infty$$

On the other hand, we have $|e^{-2\pi i z\xi}| = e^{2\pi y\xi}$.

Let Γ_R be the positively oriented boundary of the rectangle bounded by the lines $y = 0, -\xi$ and $x = \pm R$.

Let

l_1 be the line segment from	$-R$	to	R
l_2 be the line segment from	R	to	$R - \xi i$
l_3 be the line segment from	$R - \xi i$	to	$-R - \xi i$
l_4 be the line segment from	$-R - \xi i$	to	$-R$

Now on the line segment l_3, we have

$$f(x - \xi i) = e^{-\pi(x-\xi i)^2} e^{-2\pi i(x-\xi i)\xi} = e^{-\pi x^2} e^{-\pi\xi^2} \qquad \text{for } -R \le x \le R.$$

Hence,

$$\int_{l_3} f(z)\, dx = \int_{R}^{-R} e^{-\pi x^2} e^{-\pi\xi^2}\, dx = -e^{-\pi\xi^2} \int_{-R}^{R} e^{-\pi x^2}\, dx$$

Moreover,

$$\left|\int_{l_2} f(z)\, dz\right| \le \int_0^{-\xi} \left|e^{-\pi(R+iy)^2} e^{-2\pi i(R+iy)\xi}\right| dy = \int_0^{-\xi} e^{-\pi(R^2-y^2)} e^{2\pi y\xi}\, dx$$
$$\le -\xi e^{-\pi(R^2-\xi^2)} \to 0 \quad \text{as } R \to \infty$$

Similarly, $\int_{l_4} f(z)\, dz \to 0$ as $R \to \infty$. By Cauchy's integral formula, for the entire function $f(z)$, we have $\int_{\Gamma_R} f(z)\, dz = 0$ and as $R \to \infty$, we obtain

$$\int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i x\xi}\, dx = e^{-\pi\xi^2}$$

For $\xi > 0$, since $-\xi < 0$, we have

$$\int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i x\xi}\, dx = \int_{\infty}^{-\infty} e^{-\pi(-x)^2} e^{-2\pi i(-x)\xi}\, d(-x) = \int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i x(-\xi)}\, dx = e^{-\pi\xi^2}.$$

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