THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MMAT5220 Complex Analysis and Its Applications 2019-20
Week 8 Examples

1. Let $f(z) = e^z - 1$ be an entire function. Then, the zero set of f is $\{\pm 2\pi n i : n \in \mathbb{N} \cup \{0\}\}$. The function
$$T(z) = -i\frac{z-1}{z+1}$$
maps the unit circle and its interior onto $\{y = 0\} \cup \{\infty\}$ and the upper half plane $\{y > 0\}$. Hence, $f \circ T(z) = e^{-i\frac{z-1}{z+1}} - 1$ is an analytic function on the open unit disk $\{|z| < 1\}$ and it has infinitely many zeros there. Notice that the zero set of the function $f \circ T$ does not contain a limit point in the open unit disk $\{|z| < 1\}$.

2. Suppose $f(z)$ is analytic in a punctured disc $\{0 < |z - z_0| < r\}$. Suppose also that
$$|f(z)| \le A|z - z_0|^{-1+\epsilon}$$
for some $\epsilon > 0$, and all z near z_0. Show that the singularity of f at z_0 is removable.

Solution. You may write
$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$$
as a Laurent series expansion on $\{0 < |z - z_0| < r\}$, and then argue that $b_n = 0$ for all $n \ge 1$ as in Week 8 Lecture notes, together with the given inequality.

Another way is to apply the theorem. Note the function $f(z)(z - z_0)$ is an analytic function on $\{0 < |z - z_0| < r\}$ and $f(z)(z - z_0)$ is bounded for z near z_0. Indeed,
$$|f(z)(z - z_0)| \le A|z - z_0|^{\epsilon} \le A$$
for all z near z_0. By the theorem, there is some analytic function g on $\{|z - z_0| < r\}$ such that $f(z)(z - z_0) = g(z)$ in the disc punctured at z_0. Using the inequality again, we see that $g(0) = 0$ and hence $g(z) = (z - z_0)^m g_1(z)$, where g_1 is analytic on $\{|z - z_0| < r\}$ and $m \in \mathbb{N}$.

In conclusion, $f(z) = (z - z_0)^{m-1} g_1(z)$ on the punctured disc. Note that $(z - z_0)^{m-1} g_1(z)$ is analytic at z_0. It shows that the singularity of f at z_0 is removable. ◀

3. Let f be a non-constant entire function, i.e. a function analytic on $\mathbb{C}$. Show that the image of f is dense in $\mathbb{C}$.

Solution. Suppose not, there is some $w_0 \in \mathbb{C}$ and $\epsilon_0 > 0$ such that $|f(z) - w_0| \ge \epsilon_0$ for all $z \in \mathbb{C}$. Hence, $1/(f(z) - w_0)$ is an entire function bounded by $1/\epsilon_0$. By Liouville's theorem, $1/(f(z) - w_0)$ is a constant function. This contradicts to the assumption that f is a non-constant function. Therefore, the image of f is dense in $\mathbb{C}$. ◀

4. Find the residues of the following functions at 1.
   (a) $1/(z^2-1)(z+2)$; (b) $(z^3-1)(z+2)/(z^4-1)^2$

   Solution.

   (a) Notice that
   $$\frac{1}{(z^2-1)(z+2)} = \frac{1}{(z-1)} \cdot \frac{1}{(z+1)(z+2)} = \frac{\phi(z)}{z-1}.$$
   The function $\phi(z) = \frac{1}{(z+1)(z+2)}$ is analytic at $z=1$, and hence
   $$\phi(z) = \sum_{n=0}^{\infty} \frac{\phi^{(n)}(1)}{n!}(z-1)^n \qquad \text{for all } z \text{ near } 1.$$
   Since
   $$\frac{1}{(z^2-1)(z+2)} = \frac{\phi^{(0)}(1)}{0!}\frac{1}{z-1} + \sum_{n=0}^{\infty} \frac{\phi^{(n+1)}(1)}{(n+1)!}(z-1)^n,$$
   the residue of the given function at 1 is $\phi(1) = 1/6$.

   (b) Simplifying the quotient, we have
   $$f(z) = \frac{(z^2+z+1)(z+2)}{(z-1)(z^3+z^2+z+1)^2}.$$
   Note that $z=1$ is a simple pole of f. The residue of f at 1 is
   $$\lim_{z\to 1} f(z)(z-1) = \frac{(1^2+1+1)(1+2)}{(1^3+1^2+1+1)^2} = \frac{9}{16}.$$

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5. Find the value of the integral
   $$\int_C \frac{dz}{z^3(z+4)},$$
   taken counterclockwise around the circle (a) $|z|=2$; (b) $|z+2|=3$.

   Solution. Notice that the given function f has a pole of order 3 at $z=0$, and a simple pole at $z=-4$. To calculate $\operatorname*{Res}_{z=0} f(z)$, we note that
   $$\frac{1}{z+4} = \frac{1}{4}(1 - \frac{z}{4} + \frac{z^2}{16} - \frac{z^3}{64} + \cdots),$$
   Hence, $\operatorname*{Res}_{z=0} f(z) = 1/64$. On the other hand, $\operatorname*{Res}_{z=-4} f(z) = 1/(-4)^3 = -1/64$.

   (a) For the contour $|z|=2$, its interior contains the pole $z=0$ only. By Cauchy's residue theorem, the integral equals $2\pi i \operatorname*{Res}_{z=0} f(z) = \pi i/32$.

   (b) For the contour $|z+2|=3$, its interior contains the pole $z=0$ and $z=-4$. By Cauchy's residue theorem, the integral equals $2\pi i(\operatorname*{Res}_{z=0} f(z) + \operatorname*{Res}_{z=-4} f(z)) = 0$.

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