THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MMAT5220 Complex Analysis and Its Applications 2019-20
Week 10 Examples

1. Evaluate the following integrals by the method of residues:

(a) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{dx}{a+\sin^2 x},\ a\in\mathbb{R}, |a|>1,$ (b) $\displaystyle\int_0^{2\pi} \frac{\cos^2 3\theta}{5-4\cos 2\theta}\, d\theta.$

(c) $\displaystyle\int_0^{\infty} \frac{\log x}{1+x^2}\, dx$ (d) $\displaystyle\int_0^{\infty} \log(1+x^2)\frac{dx}{x^{1+\alpha}},\ 0<\alpha<2.$

You may try integration by parts for (d).

Solution. (a) Using the double angle formula, we have

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \frac{dx}{a+\sin^2 x} &= \int_0^{\frac{\pi}{2}} \frac{dx}{a+\frac{1}{2}(1-\cos 2x)} \\ &= \int_0^{\pi} \frac{d\theta}{2a+(1-\cos\theta)} && (\theta = 2x) \\ &= \frac{1}{2}\int_{-\pi}^{\pi} \frac{d\theta}{2a+(1-\cos\theta)} && \text{(even function)}\end{aligned}$$

Let $z=e^{i\theta}$ and consider the positively oriented contour $\{|z|=1\}$, we see that

$$\begin{aligned}\int_{-\pi}^{\pi} \frac{d\theta}{2a+(1-\cos\theta)} &= \int_{|z|=1} \frac{1}{2a+(1-\frac{z+z^{-1}}{2})}\frac{dz}{iz} \\ &= \int_{|z|=1} \frac{2}{(4a+2)z-z^2-1}\frac{dz}{i} \\ &= 2i\int_{|z|=1} \frac{dz}{z^2-(4a+2)z+1}\end{aligned}$$

The polynomial $z^2-(4a+2)z+1$ has two real roots

$$\alpha = 2a+1+2\sqrt{a^2+a}, \qquad \beta = 2a+1-2\sqrt{a^2+a}$$

Note that α is outside the unit circle if $a>1$, while β is outside if $a<-1$. Also, the integral $\int_0^{\frac{\pi}{2}} \frac{dx}{a+\sin^2 x}$ is nonzero in any cases. This implies that the other roots will lie inside the circle $|z|=1$. Using the residue, we have

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \frac{dx}{a+\sin^2 x} &= \begin{cases} \frac{2\pi i(2i)}{2(\beta-\alpha)} & \text{if } a>1, \\ \frac{2\pi i(2i)}{2(\alpha-\beta)} & \text{if } a<-1. \end{cases} \\ &= \frac{a}{|a|}\frac{\pi}{2\sqrt{a^2+a}}\end{aligned}$$

(b) Let $z = e^{i\theta}$ and consider the positively oriented contour $\{|z| = 1\}$, then

$$\cos 3\theta = \frac{1}{2}\left(z^3 + \frac{1}{z^3}\right), \qquad \cos 2\theta = \frac{1}{2}\left(z^2 + \frac{1}{z^2}\right).$$

That is,

$$\begin{aligned}\int_0^{2\pi} \frac{\cos^2 3\theta}{5 - 4\cos 2\theta}\, d\theta &= \int_{|z|=1} \frac{(z^3 + z^{-3})^2}{4(5 - 2(z^2 + z^{-2}))} \frac{dz}{iz} \\ &= \int_{|z|=1} \frac{(z^6 + 1)^2}{4z^5(5z^2 - 2(z^4 + 1))} \frac{dz}{i} \\ &= \frac{i}{4} \int_{|z|=1} \frac{(z^6 + 1)^2}{z^5(2z^4 - 5z^2 + 2)}\, dz\end{aligned}$$

The polynomial $2z^4 - 5z^2 + 2$ can be factorized as

$$(2z^2 - 1)(z^2 - 2) = 2(z - 1/\sqrt{2})(z + 1/\sqrt{2})(z - \sqrt{2})(z + \sqrt{2})$$

So, if we put $f(z) = \frac{(z^6+1)^2}{z^5(2z^4-5z^2+2)}$, then all the singular points of $f(z)$ inside the unit circle are $z = 0, -1/\sqrt{2}, 1/\sqrt{2}$. We would apply Cauchy's residue theorem to compute the integral $\int_{|z|=1} f(z)\, dz$. Note that

$$\operatorname*{Res}_{z=\frac{-1}{\sqrt{2}}} f(z) = \frac{(-\sqrt{2})^5((\frac{-1}{\sqrt{2}})^6 + 1)^2}{2(\frac{-2}{\sqrt{2}})(\frac{-1}{\sqrt{2}} - \sqrt{2})(\frac{-1}{\sqrt{2}} + \sqrt{2})} = -\frac{27}{16}.$$

Similarly, one also has $\operatorname*{Res}_{z=1/\sqrt{2}} f(z) = -27/16$.

For $\operatorname*{Res}_{z=0} f(z)$, we observe that

$$f(z) = \frac{z^7 + 2z}{2z^4 - 5z^2 + 2} + \frac{1}{z^5(2z^4 - 5z^2 + 2)} = h(z) + \frac{1}{2z^5}\left(\frac{1}{1 - (\frac{5}{2}z^2 - z^4)}\right),$$

where $h(z)$ is analytic at 0. Therefore, the Laurent series of $f(z)$ around $z = 0$ is

$$\begin{aligned}f(z) &= \frac{1}{2z^5}\left(1 + \frac{5}{2}z^2 - z^4 + (\frac{5}{2}z^2 - z^4)^2 + \cdots\right) + \cdots \\ &= \frac{1}{2z^5}(1 + \frac{5}{2}z^2 - z^4 + \frac{25}{4}z^4 + \cdots) + \cdots \quad \text{(up to } z^{-1}\text{)}\end{aligned}$$

Hence, we have $\operatorname*{Res}_{z=0} f(z) = \dfrac{21}{8}$.

By residue theorem,

$$\int_{|z|=1} f(z)\, dz = 2\pi i(-\frac{27}{16} - \frac{27}{16} + \frac{21}{8}) = -\frac{3}{2}\pi i$$

and $\int_0^{2\pi} \frac{\cos^2 3\theta}{5-4\cos 2\theta}\, d\theta = 3\pi/8$.

(c) Consider an indented contour $\Gamma_{\epsilon,R}$ composed of two upper semicircles and two line segments, one line segment from $-R$ to $-\epsilon$, and the other from ϵ to R. The two semicircles are centered at 0 with radii ϵ and R respectively. We assume that R is large and ϵ is small. Also, we consider the function

$$f(z) = \frac{\log z}{1+z^2} \quad \text{with chosen branch } -\frac{\pi}{2} < \theta \le \frac{3\pi}{2}.$$

The function $f(z)$ is analytic on and inside $\Gamma_{\epsilon,R}$ except at the point $z = i$. By Cauchy's residue theorem,

$$\int_{\Gamma_{\epsilon,R}} f(z)\,dz = 2\pi i \operatorname*{Res}_{z=i} f(z) = 2\pi i(\frac{1}{2i}\log i) = \frac{i\pi^2}{2}.$$

On the other hand,

$$\int_{\Gamma_{\epsilon,R}} f(z)\,dz = \int_{-R}^{-\epsilon} f(z)\,dz + \int_{-C_\epsilon^+} f(z)\,dz + \int_{\epsilon}^{R} f(z)\,dz + \int_{C_R^+} f(z)\,dz,$$

where C_ϵ^+, C_R^+ are upper semicircles of radii ϵ and R oriented in counterclockwise direction. Now, note that

$$\begin{aligned} \int_{-R}^{-\epsilon} f(z)\,dz + \int_{\epsilon}^{R} f(z)\,dz &= \int_{\epsilon}^{R} \frac{\log(-z)}{1+z^2}\,dz + \int_{\epsilon}^{R} \frac{\log z}{1+z^2}\,dz \\ &= 2\int_{\epsilon}^{R} \frac{\log z}{1+z^2}\,dz + \int_{\epsilon}^{R} \frac{\pi i}{1+z^2}\,dz \end{aligned}$$

Applying residue theorem to the function $1/(1+z^2)$ on a contour composed of an upper semicircle and the diameter with large radius, we can conclude that

$$\int_0^\infty \frac{1}{1+z^2}\,dz = \frac{1}{2}\int_{-\infty}^{\infty} \frac{1}{1+z^2}\,dz = \frac{1}{2}(2\pi i)(\frac{1}{2i}) = \frac{\pi}{2}.$$

Hence,

$$\lim_{\substack{\epsilon\to 0\\ R\to\infty}} \left(\int_{-R}^{-\epsilon} f(z)\,dz + \int_{\epsilon}^{R} f(z)\,dz\right) = 2\int_0^\infty \frac{\log z}{1+z^2}\,dz + \frac{i\pi^2}{2}.$$

Also, using L'Hôpital's rule, we have

$$\begin{aligned} \left|\int_{-C_\epsilon^+} f(z)\,dz\right| &\le \pi\epsilon\frac{|\log\epsilon| + \frac{3\pi}{2}}{1-\epsilon^2} \to 0 \quad \text{as } \epsilon \to 0 \\ \left|\int_{C_R^+} f(z)\,dz\right| &\le \pi R\frac{\log R + \frac{3\pi}{2}}{R^2-1} \to 0 \quad \text{as } R \to \infty \end{aligned}$$

In conclusion, we have $\int_0^\infty \frac{\log x}{1+x^2}\,dx = 0$. Rather than using residues, one may substitute $y = 1/x$ to obtain

$$\int_1^\infty \frac{\log x}{1+x^2}\,dx = \int_0^1 \frac{-\log y}{1+\frac{1}{y^2}}\,\frac{dy}{y^2} = -\int_0^1 \frac{\log y}{1+y^2}\,dy,$$

and draw the same conclusion.

(d) Doing integration by parts, we have

$$\begin{aligned}\int_0^\infty \log(1+x^2)\,\frac{dx}{x^{1+\alpha}} &= \int_0^\infty \frac{1}{-\alpha}\log(1+x^2)\,dx^{-\alpha}\\ &= \frac{-x^{-\alpha}}{\alpha}\log(1+x^2)|_{x=0}^{\infty} + \frac{1}{\alpha}\int_0^\infty x^{-\alpha}\frac{2x\,dx}{1+x^2}\\ &= \frac{2}{\alpha}\int_0^\infty \frac{x^{1-\alpha}}{1+x^2}\,dx\end{aligned}$$

It can be checked that we have $\lim_{x\to 0}\frac{\log(1+x^2)}{x^\alpha} = 0$ and $\lim_{x\to\infty}\frac{\log(1+x^2)}{x^\alpha} = 0$ by L'Hôptial's rule. Now, we apply the contour described in part (c), and let

$$f(z) = \frac{z^{1-\alpha}}{1+z^2} = \frac{e^{(1-\alpha)\log z}}{1+z^2},$$

where the branch of the $\log$ function is chosen to be $-\frac{\pi}{2} < \arg z \le \frac{3\pi}{2}$. Therefore, the function $f(z)$ is analytic on and inside the contour $\Gamma_{\epsilon,R}$ except at the point $z = i$. By residue theorem, we see that

$$\int_{\Gamma_{\epsilon,R}} f(z)\,dz = 2\pi i \operatorname*{Res}_{z=i} f(z) = 2\pi i\left(\frac{e^{(1-\alpha)\log i}}{2i}\right) = \pi e^{\frac{\pi}{2}i - \frac{\pi\alpha}{2}i} = \pi i e^{\frac{-i\pi\alpha}{2}}$$

On the other hand,

$$\int_{\Gamma_{\epsilon,R}} f(z)\,dz = \int_{-R}^{-\epsilon} f(z)\,dz + \int_{-C_\epsilon^+} f(z)\,dz + \int_\epsilon^R f(z)\,dz + \int_{C_R^+} f(z)\,dz,$$

where C_ϵ^+, C_R^+ are upper semicircles of radii ϵ and R oriented in counterclockwise direction. We will calculate the integrals over the line segments $(-R,-\epsilon)$ and (ϵ, R) respectively, and then claim that the integrals over the semicircles will tend to 0 as $\epsilon \to 0$ and $R \to \infty$.

$$\begin{aligned}\int_\epsilon^R f(z)\,dz &= \int_\epsilon^R \frac{x^{1-\alpha}}{1+x^2}\,dx\\ \int_{-R}^{-\epsilon} f(z)\,dz &= \int_{-R}^{-\epsilon} \frac{e^{(1-\alpha)\log z}}{1+z^2}\,dz\\ &= \int_\epsilon^R \frac{e^{(1-\alpha)\log(-x)}}{1+x^2}\,dx\\ &= \int_\epsilon^R \frac{e^{(1-\alpha)(\log x + i\pi)}}{1+x^2}\,dx\\ &= -e^{-i\pi\alpha}\int_\epsilon^R \frac{x^{1-\alpha}}{1+x^2}\,dx\end{aligned}$$

Moreover,

$$\begin{aligned}\left|\int_{-C_\epsilon^+} f(z)\,dz\right| &\le \pi\epsilon\frac{\epsilon^{1-\alpha}}{1-\epsilon^2} \to 0 \quad \text{as } \epsilon \to 0\\ \left|\int_{C_R^+} f(z)\,dz\right| &\le \pi R\frac{R^{1-\alpha}}{R^2-1} \to 0 \quad \text{as } R \to \infty\end{aligned}$$

The convergence is due to the observation $0 < 2-\alpha < 2$. In conclusion, as $\epsilon \to 0$ and $R \to \infty$, we obtain

$$\pi i e^{\frac{-i\pi\alpha}{2}} = (1-e^{-i\pi\alpha})\int_0^\infty \frac{x^{1-\alpha}}{1+x^2}\,dx$$

$$\int_0^\infty \frac{x^{1-\alpha}}{1+x^2}\,dx = \frac{\pi i}{e^{\frac{i\pi\alpha}{2}} - e^{\frac{-i\pi\alpha}{2}}} = \frac{\pi}{2\sin\frac{\pi\alpha}{2}}$$

Therefore,

$$\int_0^\infty \log(1+x^2)\,\frac{dx}{x^{1+\alpha}} = \frac{\pi}{\alpha\sin\frac{\pi\alpha}{2}}$$

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2. Prove that

$$\int_0^\infty \sin(x^2)\,dx = \int_0^\infty \cos(x^2)\,dx = \frac{\sqrt{2\pi}}{4}.$$

These are the **Fresnel integrals**. [Hint: Integrate the function e^{iz^2} over the path: from 0 to R, and then from R to $Re^{i\frac{\pi}{4}}$ along the minor arc of circle $|z| = R$, and back to 0 through the straight line. Recall that $\int_\infty^\infty e^{-x^2}\,dx = \sqrt{\pi}$.]

Solution. Let l_1 be the line segment from 0 to R, C_R be the minor arc described in the hint, and l_2 be the line segment from $Re^{i\frac{\pi}{4}}$ to 0. Let Γ_R be the positively oriented contour composed of l_1, l_2 and C_R. Note that the function $f(z) = e^{iz^2}$ is entire. In particular, by Cauchy-Goursat theorem, we have

$$\int_{\Gamma_R} f(z)\,dz = 0.$$

On the other hand, on the line segment l_1, we have

$$\int_{l_1} f(z)\,dz = \int_0^R e^{ix^2}\,dx = \int_0^R \cos(x^2)\,dx + i\int_0^R \sin(x^2)\,dx.$$

and on l_2, we have

$$\begin{aligned}\int_{l_2} f(z)\,dz &= \int_R^0 e^{i(re^{\frac{i\pi}{4}})^2} e^{\frac{i\pi}{4}}\,dr \\ &= -e^{\frac{i\pi}{4}}\int_0^R e^{i(re^{\frac{i\pi}{4}})^2}\,dr \\ &= -e^{\frac{i\pi}{4}}\int_0^R e^{ir^2 e^{\frac{i\pi}{2}}}\,dr \\ &= -e^{\frac{i\pi}{4}}\int_0^R e^{-r^2}\,dr\end{aligned}$$

Moreover, on the arc C_R, we claim that the integral goes to 0 as $R \to \infty$. Recall that $\sin x \geq \frac{2x}{\pi}$ on $[0, \frac{\pi}{2}]$. (see Week 9 Lecture) Now,

$$\begin{aligned}\left|\int_{C_R} f(z)\, dz\right| &= \left|\int_0^{\frac{\pi}{4}} e^{i(Re^{i\theta})^2} Re^{i\theta}\, id\theta\right| \\ &\leq \int_0^{\frac{\pi}{4}} \left|e^{iR^2(\cos 2\theta + i\sin 2\theta)}\right| R\, d\theta \\ &= R\int_0^{\frac{\pi}{4}} e^{-R^2 \sin 2\theta}\, d\theta \\ &\leq R\int_0^{\frac{\pi}{4}} e^{-\frac{4R^2}{\pi}\theta}\, d\theta \\ &= \frac{\pi}{4R}\left(1 - e^{-R^2}\right) \to 0 \quad \text{as } R \to \infty.\end{aligned}$$

To conclude, as $R \to \infty$, we obtain

$$\int_0^\infty \cos(x^2)\, dx = \int_0^\infty \sin(x^2)\, dx = \frac{1}{\sqrt{2}} \frac{\sqrt{\pi}}{2} = \frac{\sqrt{2\pi}}{4}.$$

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3. Let U be a simply connected domain and $z_0 \in U$. Suppose h is an analytic function on U and $h(z) \neq 0$ for all $z \in U$. Put $f(z) = (z - z_0)^m h(z)$ for some $m \in \mathbb{Z}$. If γ is a closed contour such that $z_0 \notin \gamma$, prove that

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\, dz = n(\gamma, z_0)m.$$

where $n(\gamma, z_0)$ is the winding number of γ around z_0.

Solution. Notice that

$$\begin{aligned} f'(z) &= m(z - z_0)^{m-1}h(z) + (z - z_0)^m h'(z) \\ \frac{f'(z)}{f(z)} &= \frac{m}{z - z_0} + \frac{h'(z)}{h(z)}\end{aligned}$$

By Extended Cauchy-Goursat theorem (Week 4 Lecture) and the definition of winding number, we have

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\, dz = n(\gamma, z_0)m + 0 = n(\gamma, z_0)m.$$

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4. Determine the number of zeros of the polynomial

$$z^{87} + 36z^{57} + 71z^4 + z^3 - z + 1$$

inside the circle

(a) of radius 1,

(b) of radius 2, centered at the origin.

(c) Determine the number of zeros of the polynomial

$$2z^5 - 6z^2 + z + 1 = 0$$

in the annulus $1 \le |z| \le 2$.

Solution. (a) Let $f(z) = 71z^4$ and $g(z) = z^{87} + 36z^{57} + z^3 - z + 1$. Then, both f and g are entire functions. Also,

$$|g(z)| \le |z|^{87} + 36|z|^{57} + |z|^3 + |z| + 1 = 40 < 71 = |f(z)| \quad \text{on } |z| = 1.$$

By Rouché's theorem, the polynomial $f + g$ and f have the same number of zeros inside $|z| = 1$, which is equal to 4.

(b) Let $f(z) = z^{87}$ and $g(z) = 36z^{57} + 71z^4 + z^3 - z + 1$. Then, both f and g are entire functions. Also, on the circle $|z| = 2$, we have

$$\begin{aligned} |g(z)| &\le 36|z|^{57} + 71|z|^4 + |z|^3 + |z| + 1 \\ &\le 2^6 \cdot 2^{57} + 2^7 \cdot 2^4 + 2^3 + 2 + 2 \\ &\le 2^6 \cdot 2^{57} \cdot 5 \\ &\le 2^{66} < 2^{87} = |f(z)| \end{aligned}$$

By Rouché's theorem, the polynomial $f + g$ and f have the same number of zeros inside $|z| = 2$, which is equal to 87.

(c) Let $f_1(z) = 2z^5$ and $g_1(z) = -6z^2 + z + 1$. Then, both f and g are entire functions. Also, on the circle $|z| = 2$, we have

$$|g_1(z)| \le 6|z|^2 + |z| + 1 = 27 < 64 = |f_1(z)|$$

By Rouché's theorem, the polynomial $f_1 + g_1$ and f_1 have the same number of zeros inside $|z| = 2$, which is equal to 5. Recall that the inequality $|f_1(z) + g_1(z)| \ge |f_1(z)| - |g_1(z)| > 0$ automatically tells us that both $f_1 + g_1$ and f_1 have no zero on the circle $\{|z| = 2\}$.

On the other hand, we put $f_2(z) = -6z^2$ and $g_2(z) = 2z^5 + z + 1$. Using Rouché's theorem again, we can show that the number of zeros of $f_2 + g_2$ inside the circle $|z| = 1$ is 2. Therefore, the number of zeros of the polynomial in the annulus $\{1 \le |z| \le 2\}$ is $5 - 2 = 3$.

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5. Let f be analytic on the closed unit disc $\overline{D}$.

(a) Assume that $|f(z)| = 1$ if $|z| = 1$, and f is not constant. Prove that the image of f contains the closed unit disc.

(b) Assume that there exists some point $z_0 \in D$ such that $|f(z_0)| < 1$, and that $|f(z)| \ge 1$ if $|z| = 1$. Prove that $f(D)$ contains the open unit disc

Solution. (a) Recall that by Maximum Modulus Principle, we can deduce that f must attain 0 inside the circle $\{|z| = 1\}$. (see Week 5 Examples) Now, let $|w_0| < 1$, notice that

$$|-w_0| < 1 = |f(z)| \quad \text{for } |z| = 1.$$

By Rouché's theorem, $f(z)$ and $f(z) - w_0$ have the same number of zeros inside the unit circle. This shows that $f(z_0) = w_0$ for some $|z_0| < 1$. Since w_0 is arbitrary, $D \subseteq f(\overline{D})$. The continuity of f further tells us that $\overline{D} \subseteq f(\overline{D})$.

(b) Naively, if we put $\gamma = \{|z| = 1\}$, then the assumption tells us that $f(z) - f(z_0)$ attains zeros inside the unit circle. Since the function $f(z)$ is analytic, it has no poles inside the contour γ. By argument principle, the contour $f(\gamma) - f(z_0)$ circulate the point $z = 0$ at least once and hence, $f(\gamma)$ would enclose the point $f(z_0)$. However, $|f(\gamma)| \geq 1$. A picture will show that every point inside the unit circle is enclosed by $f(\gamma)$. Using the argument principle again, we see that $f(D)$ contains the unit disc.

To argue this formally, note

$$|f(z)| \geq 1 > |-f(z_0)| \quad \text{for } |z| = 1.$$

Rouché's theorem implies that $f(z)$ and $f(z) - f(z_0)$ have the same number of zeros inside the unit circle. In particular, $f(z) = 0$ for some $|z| < 1$. Now let any $|w_0| < 1$ and notice that

$$|f(z)| \geq 1 > |-w_0| \quad \text{for } |z| = 1.$$

By Rouché's theorem again, we can conclude that $f(z) = w_0$ for some $|z| < 1$. This shows that $f(D)$ contains the open unit disc.

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