THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MMAT5220 Complex Analysis and Its Applications 2019-20
Homework 5
Due Date: 16th April 2020

Compulsory Part

1. Find the residue at $z = 0$ of the following functions:

   (a) $\dfrac{1}{z + z^2}$;

   (b) $z \cos\left(\dfrac{1}{z}\right)$.

   Solution.

   (a) Let $f(z) = 1/(z + z^2)$ and $h(z) = 1/(1 + z)$. Notice that $h(z)$ is analytic at $z = 0$ and
   $$f(z) = \frac{1}{z}h(z)$$
   Therefore, we have $\operatorname*{Res}_{z=0} f(z) = h(0) = 1$.

   (b) Let $f(z) = z\cos(1/z)$. To find the residue of f at $z = 0$, we may consider the Laurent series of $\cos(1/z)$ around $z = 0$. The coefficient of the term $1/z^2$ in the series expansion will be the required residue. Note that
   $$\cos\left(\frac{1}{z}\right) = 1 - \frac{1}{2!}\left(\frac{1}{z^2}\right) + \frac{1}{4!}\left(\frac{1}{z^4}\right) + \cdots$$
   Hence, $\operatorname*{Res}_{z=0} f(z) = -1/2$.

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2. For each of the following functions, find all its isolated singular points, write down their principal parts, classify their types, and compute the residues:

   (a) $\dfrac{z-1}{z^2 - 5z + 4}$;

   (b) $\sin\left(\dfrac{2}{z}\right)$;

   (c) $\dfrac{z+1}{\cos z}$.

   Solution. (a) Notice that $z^2 - 5z + 4 = (z-4)(z-1)$ and hence $\{1, 4\}$ are all isolated singular points of the given function. Moreover,
   $$\frac{z-1}{z^2-5z+4} = \frac{1}{z-4} \qquad \text{for } z \neq 1 \text{ or } 4$$
   So, $z = 1$ is a removable singularity, the function has no principal part at $z = 1$ and the residue at 1 is 0; $z = 4$ is a simple pole, the principal part is $1/(z-4)$, and the residue is 1.

(b) Clearly, $z = 0$ is the only singular point. Moreover, for $z \neq 0$,

$$\sin\left(\frac{2}{z}\right) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)!}\left(\frac{2}{z}\right)^{2n-1}.$$

It is an essental singularity and the principal part is the whole series given above. The residue of the function at 0 is 2. (corresponding to $n = 1$ in the expansion above.)

(c) Let $f(z) = (z+1)/\cos z$. Notice that $\cos z = \frac{e^{iz}+e^{-iz}}{2} = 0$ iff $e^{2iz} = -1$. That is, $z = (n-1/2)\pi$ for any $n \in \mathbb{Z}$. All isolated singular points of $f(z)$ are $\{(n-1/2)\pi : n \in \mathbb{Z}\}$. Fix any $n \in \mathbb{Z}$, notice that

$$\begin{aligned}
\cos z &= \cos(z - (n-\frac{1}{2})\pi + (n-\frac{1}{2})\pi) \\
&= (-1)^n \sin(z - (n-\frac{1}{2})\pi) \\
&= (-1)^n \left(z - (n-\frac{1}{2})\pi - \frac{1}{3!}(z-(n-\frac{1}{2})\pi)^3 + \frac{1}{5!}(z-(n-\frac{1}{2})\pi)^5 + \cdots \right)
\end{aligned}$$

Hence, $\frac{1}{\cos z}$ has a simple pole of order 1 at $z = (n - \frac{1}{2})\pi$, with residue $1/(-1)^n$. Since $z+1$ is an entire function, the principal part of $f(z)$ at $(n-1/2)\pi$ is $(-1)^n(n\pi - \pi/2+1)(z-(n-1/2)\pi)^{-1}$, and the residue is $(-1)^n(n\pi - \pi/2 + 1)$.

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3. Use residues to evaluate the integral $\displaystyle\int_{|z|=3} \frac{2z-3}{z(z+1)}dz$.

Solution. Let $f(z) = \frac{2z-3}{z(z+1)}$. All singular points inside the circle $\{|z| = 3\}$ are $0, -1$. So by Cauchy's residue theorem,

$$\int_{|z|=3} \frac{2z-3}{z(z+1)}dz = 2\pi i(\operatorname*{Res}_{z=0} f(z) + \operatorname*{Res}_{z=-1} f(z)) = 2\pi i(-3+5) = 4\pi i.$$

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4. Suppose that q is analytic and has a zero of order 1 at z_0. Show that $f = 1/q^2$ has a pole of order 2 at z_0 with residue given by

$$\text{Res}_{z=z_0} f(z) = -\frac{q''(z_0)}{(q'(z_0))^3}.$$

Solution. Since q is analytic and has a zero of order 1 at z_0, we have $q'(z_0) \neq 0$ and

$$\begin{aligned}
q(z) &= q'(z_0)(z-z_0) + \frac{q''(z_0)}{2}(z-z_0)^2 + \frac{q'''(z_0)}{3!}(z-z_0)^3 + \cdots \quad \text{for } z \text{ near } z_0. \\
&= q'(z_0)(z-z_0)(1 + \frac{q''(z_0)}{2q'(z_0)}(z-z_0) + \frac{q'''(z_0)}{6q'(z_0)}(z-z_0)^2 + \cdots) \\
&= q'(z_0)(z-z_0)(1-h(z)),
\end{aligned}$$

where

$$h(z) = -\frac{q''(z_0)}{2q'(z_0)}(z-z_0) - \frac{q'''(z_0)}{6q'(z_0)}(z-z_0)^2 + \cdots$$

is analytic at z_0 and has a zero of order not less than 1 at z_0. Therefore, we have

$$\frac{1}{q(z)} = \frac{1}{q'(z_0)(z-z_0)}(1 + h(z) + h(z)^2 + \cdots)$$
$$\frac{1}{q(z)^2} = \frac{1}{q'(z_0)^2(z-z_0)^2}(1 + 2h(z) + 3h(z)^2 + \cdots)$$

Recall that $h(z)$ has a zero of order at least 1 at z_0, the residue of f at z_0 comes from the term $2h(z)$, which is $\frac{1}{q'(z_0)^2}\frac{-2q''(z_0)}{2q'(z_0)} = -q''(z_0)/q'(z_0)^3$.

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5. For any $N > 0$, let γ_N be the positively oriented boundary of the square bounded by the lines $x = \pm(N+\frac{1}{2})\pi$ and $y = \pm(N+\frac{1}{2})\pi$.

   (a) Show that

$$\int_{\gamma_N} \frac{dz}{z^2 \sin z} = 2\pi i\left(\frac{1}{6} + 2\sum_{n=1}^{N}\frac{(-1)^n}{n^2\pi^2}\right).$$

   (b) Using (a), show that

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$$

   by estimating $\left|\int_{\gamma_N}\frac{dz}{z^2\sin z}\right|$ in terms of N.

Solution. (a) Let $f(z) = 1/(z^2\sin z)$. All singular points of $f(z)$ inside γ_N are $\{n\pi : n \in \mathbb{Z}, -N \le n \le N\}$. For $n \neq 0$, $f(z)$ has a simple pole there and hence,

$$\operatorname*{Res}_{z=n\pi} f(z) = \lim_{z\to n\pi}\frac{z-n\pi}{z^2\sin z} = \lim_{y\to 0}\frac{y}{(y+n\pi)^2\sin(y+n\pi)} = \lim_{y\to 0}\frac{(-1)^n y}{(y+n\pi)^2\sin y} = \frac{(-1)^n}{n^2\pi^2}$$

using $(y = z - n\pi)$. On the other hand, the residue of $f(z)$ at $z = 0$ is the coefficient of the term z in the Laurent series expansion of $1/\sin z$. Note that

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots$$
$$= z(1 - (\frac{z^2}{3!} - \frac{z^4}{5!} + \cdots))$$
$$\frac{1}{\sin z} = \frac{1}{z}(1 + (\frac{z^2}{3!} - \frac{z^4}{5!} + \cdots) + (\frac{z^2}{3!} - \frac{z^4}{5!} + \cdots)^2 + \cdots)$$

Therefore, the coefficient of z in the Laurent series expansion is $1/3! = 1/6$. By Cauchy's residue theorem, we have

$$\int_{\gamma_N}\frac{dz}{z^2\sin z} = 2\pi i\left(\sum_{n=-N}^{N}\operatorname*{Res}_{z=n\pi} f(z)\right) = 2\pi i\left(\frac{1}{6} + 2\sum_{n=1}^{N}\frac{(-1)^n}{n^2\pi^2}\right).$$

(b) Write $z = x + iy$, for $x = \pm(N + \frac{1}{2})\pi$, we have $|\sin z| = |(-1)^N \cos(iy)| = \cosh(y) \geq 1$. For $y = (N + \frac{1}{2})\pi$, we have

$$|\sin z| = \left|\frac{e^{ix}e^{-y} - e^{-ix}e^{y}}{2i}\right| \geq \frac{|e^{-ix}e^{y}| - |e^{ix}e^{-y}|}{2} \geq \frac{e^{y} - 1}{2}$$

For $y = -(N + \frac{1}{2})\pi$, we have

$$|\sin z| = \left|\frac{e^{ix}e^{-y} - e^{-ix}e^{y}}{2i}\right| \geq \frac{|e^{-ix}e^{-y}| - |e^{ix}e^{y}|}{2} \geq \frac{e^{-y} - 1}{2}$$

Therefore, $|\sin z| \geq 1$ for every $z \in \gamma_N$ and $N \geq 2$.

We may now estimate the integral for $N \geq 2$ that

$$\left|\int_{\gamma_N} \frac{dz}{z^2 \sin z}\right| \leq \frac{8(N + \frac{1}{2})\pi}{(N + \frac{1}{2})^2 \pi^2} \to 0 \quad \text{as } N \to \infty.$$

Using (a) and taking limit $N \to \infty$, we have $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$.

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Optional Part

1. Find the residue at $z = 0$ of the following functions:

   (a) $\dfrac{\cot z}{z^4}$;

   (b) $\dfrac{z^3 + 2z + 1}{z^2(z+1)}$.

Solution. (a) It suffices to find the coefficient of z^3 in the Laurent series of $\cot z$. Notice that in Q5(a) the Laurent series of $1/\sin z$ is

$$\begin{aligned}\frac{1}{\sin z} &= \frac{1}{z}(1 + (\frac{z^2}{3!} - \frac{z^4}{5!} + \cdots) + (\frac{z^2}{3!} - \frac{z^4}{5!} + \cdots)^2 + \cdots) \\ &= \frac{1}{z} + \frac{z}{6} + \frac{7z^3}{360} + \cdots\end{aligned}$$

Hence, we have

$$\begin{aligned}\cot z &= \cos z\left(\frac{1}{\sin z}\right) \\ &= (1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots)(\frac{1}{z} + \frac{z}{6} + \frac{7z^3}{360} + \cdots) \\ &= \frac{1}{z} + \frac{z}{6} + \frac{7z^3}{360} - \frac{z}{2} - \frac{z^3}{12} + \frac{z^3}{4!} + \cdots\end{aligned}$$

Therefore, the required residue is $7/360 - 1/12 + 1/24 = -1/45$.

(b) The residue at $z = 0$ is the coeffient of z in the Laurent series of $(z^3+2z+1)/(z+1)$. The Laurent series is given by

$$(z^3 + 2z + 1)(1 - z + z^2 + \cdots) = 1 + 2z + z^3 - z - 2z^2 - z^4 + \cdots$$

Hence, the residue is 1.

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2. For each of the following functions, find all its isolated singular points, write down their principal parts, classify their types, and compute the residues:

   (a) $\dfrac{\sin 3z}{z}$;

   (b) $\dfrac{z^2}{2 - \sqrt{z}}$, where the principal branch is taken for $\sqrt{z}$.

Solution. (a) Clearly, $z = 0$ is the only singular points of the given function. Moreover, since both functions $\sin 3z$ and z have zero of order 1 at $z = 0$, this is a removable singularity. It has no principal part and the residue is 0.

(b) If the principal branch is taken, then $\sqrt{z}$ is not analytic if and only if z is a non-postive real number. These singular points are not isolated. $z = 4$ is the only isolated singular point of the given function. Moreover,

$$\lim_{z\to 4} \frac{1}{2-\sqrt{z}}(z-4) = \lim_{z\to 4} -2-\sqrt{z} = -4.$$

This shows that $z = 4$ is a simple pole of the function $\frac{1}{2-\sqrt{z}}$ with residue -4. The residue of $z^2/(2-\sqrt{z})$ at $z = 4$ is $4^2(-4) = -64$.
On the other hand, we may find the Laurent series expansion of $\sqrt{z}$ at $z = 4$. Since $\sqrt{z} = \sqrt{4+(z-4)} = 2\sqrt{1+(z-4)/4}$, if the principal branch is taken, then

$$\sqrt{z} = 2\left(1+\frac{1}{2}\left(\frac{z-4}{4}\right)+\frac{(\frac{1}{2})(-\frac{1}{2})}{2!}\left(\frac{z-4}{4}\right)^2+\cdots\right)$$

$$2-\sqrt{z} = -\frac{z-4}{4}+\frac{(z-4)^2}{64}+\cdots$$

This also shows that $\operatorname*{Res}_{z=4} \dfrac{1}{2-\sqrt{z}} = -4$.

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3. Use residues to evaluate the integral $\displaystyle\int_{|z|=3} \frac{z^3}{4+z^2}dz$.

Solution. Let $f(z) = z^3/(4+z^2)$. The singular points of $f(z)$ inside the circle $\{|z| = 3\}$ are $\pm 2i$. Notice that $\operatorname*{Res}_{z=2i} f(z) = (2i)^3/(2(2i)) = -2$ and $\operatorname*{Res}_{z=-2i} f(z) = (-2i)^3/(2(-2i)) = -2$. Using Cauchy's residue theorem, we have

$$\int_{|z|=3} \frac{z^3}{4+z^2}dz = 2\pi i(\operatorname*{Res}_{z=2i} f(z) + \operatorname*{Res}_{z=-2i} f(z)) = -8\pi i.$$

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4. Let $a_1, a_2, \ldots, a_n$ be *distinct* complex numbers. Let γ be a circle around a_1 such that γ and its interior do not contain a_j for $j > 1$. Let $f(z) = (z-a_1)(z-a_2)\ldots(z-a_n)$. Find $\displaystyle\int_\gamma \frac{dz}{f(z)}$.

Solution. The only singular point of $1/f$ inside the circle γ is a_1. By Cauchy's integral formual,

$$\int_\gamma \frac{dz}{f(z)} = \frac{2\pi i}{(a_1-a_2)(a_1-a_3)\ldots(a_1-a_n)}.$$

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