THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MMAT5220 Complex Analysis and Its Applications 2019-20
Homework 2
Due Date: 5th March 2020

Compulsory Part

1. Suppose that $f(z)$ is differentiable at z_0, where $z_0 = r_0 e^{i\theta_0} \neq 0$. Show that the derivative $f'(z_0)$ can be written as
$$f'(z_0) = e^{-i\theta_0}(u_r + iv_r)$$
or
$$f'(z_0) = \frac{-i}{z_0}(u_\theta + iv_\theta),$$
where all the partial derivatives are evaluated at (r_0, θ_0).

Solution. Recall the parametrizaton $\varphi(r,\theta) = (r\cos\theta, r\sin\theta) = (x,y)$ for $0 < r < \infty$ and $0 < \theta \leq 2\pi$. If we put $g(r,\theta) = f\circ\varphi(r,\theta)$, then by chain rule, we have $Dg = DfD\varphi$, i.e.
$$\begin{pmatrix} u_r & u_\theta \\ v_r & v_\theta \end{pmatrix} = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$
Also apply the Cauchy-Riemann equations, we have
$$\begin{aligned} u_r + iv_r &= u_x\cos\theta + u_y\sin\theta + i(v_x\cos\theta + v_y\sin\theta) \\ &= u_x\cos\theta - v_x\sin\theta + i(v_x\cos\theta + u_x\sin\theta) \\ &= u_x(\cos\theta + i\sin\theta) + v_x(-\sin\theta + i\cos\theta) \\ &= u_x e^{i\theta} + iv_x e^{i\theta} \\ &= f'(z)e^{i\theta} \end{aligned}$$
This verifies the equation $f'(z_0) = e^{-i\theta_0}(u_r + iv_r)$. The other equation can be verified similarly.
$$\begin{aligned} u_\theta + iv_\theta &= u_x(-r\sin\theta) + u_y(r\cos\theta) + i(v_x(-r\sin\theta) + v_y(r\cos\theta)) \\ &= u_x(-r\sin\theta) - v_x(r\cos\theta) + i(v_x(-r\sin\theta) + u_x(r\cos\theta)) \\ &= u_x(-r\sin\theta + ir\cos\theta) + v_x(-r\cos\theta - ir\sin\theta) \\ &= iu_x z - v_x z \\ &= iz(u_x + iv_x) = izf'(z) \end{aligned}$$
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2. Consider the following function
$$f(z) = \begin{cases} (1+i)\frac{\mathrm{Im}(z^2)}{|z|^2} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0. \end{cases}$$

(a) Show that the Cauchy-Riemann equations are satisfied at $z = 0$.

(b) Is $f(z)$ differentiable at $z = 0$?

Solution.

(a) From the definition of f, we have

$$u(x, y) = v(x, y) = \begin{cases} \frac{2xy}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

Notice that

$$\partial_x u(0, 0) = \lim_{x\to 0} \frac{u(x, 0) - u(0, 0)}{x - 0} = \lim_{x\to 0} \frac{0}{x} = 0.$$

Similarly, we have $\partial_y u(0, 0) = \partial_x v(0, 0) = \partial_y v(0, 0) = 0$. Therefore, the Cauchy-Riemann equations are satisfied at $z = 0$.

(b) The function $f(z)$ is not differentiable at $z = 0$, because it is not continuous at $z = 0$. To see this, if $(x, y) = (t, t)$ for some real number $t \neq 0$, then $f(x, y) = \frac{2t^2}{2t^2} = 1$. For any $\delta > 0$, there is some $z \in \mathbb{C}$ with $|z| < \delta$, but $|f(z) - f(0)| \geq 1$, say $z = (1 + i)\frac{\delta}{2\sqrt{2}}$. From above, we see that $f(z) = 1$. On the other hand, $|z| = \sqrt{2}\frac{\delta}{2\sqrt{2}} = \frac{\delta}{2} < \delta$. This completes the proof.

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3. Let γ be the unit circle $\{z \in \mathbb{C} : |z| = 1\}$ in the counterclockwise direction. Evaluate the integral $\displaystyle\int_\gamma z^m \bar{z}^n dz$ for any $m, n \in \mathbb{Z}$.

Solution. Parametrize γ by $\gamma(t) = e^{it}$ for $0 \leq t \leq 2\pi$. Then, $z = e^{it}$, $\bar{z} = e^{-it}$ and $dz = ie^{it}dt$. The integral can be written as

$$\begin{aligned} \int_\gamma z^m \bar{z}^n dz &= \int_0^{2\pi} e^{imt} e^{-int} ie^{it} dt \\ &= i \int_0^{2\pi} e^{i(m-n+1)t} dt \\ &= \begin{cases} 2\pi i & \text{if } m - n + 1 = 0; \\ \frac{1}{m-n+1} e^{i(m-n+1)t}\Big|_{t=0}^{2\pi} & \text{otherwise.} \end{cases} \\ &= \begin{cases} 2\pi i & \text{if } m - n = -1; \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

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4. Evaluate the integral $\displaystyle\int_\gamma z^2 dz$, if

(a) γ is a straight line segment from $z = 2$ to $z = 2i$;

(b) γ is the major arc of the circle $\{z \in \mathbb{C} : |z| = 2\}$ from $z = 2$ to $z = 2i$.

Solution. Notice that z^2 is an entire function with an antiderivative $F(z) = \frac{z^3}{3}$. Therefore, the integral depends only on the end points of the contour. For both (a) and (b), the integral equals $F(2i) - F(2) = -\frac{8}{3}(i+1)$. (see Week 3 Lecture notes) ◀

5. Let γ be the arc of the circle $\{z \in \mathbb{C} : |z| = 2\}$ from $z = 2$ to $z = 2i$ that lies in the first quadrant. Show that
$$\left| \int_\gamma \frac{dz}{z^2 - 1} \right| \leq \frac{\pi}{3}.$$

Solution. For any $z \in \gamma$, we have $|z| = 2$. In particular, $|z^2 - 1| \geq |z|^2 - 1 = 3$. Moreover, γ is a quarter of the circle $\{z \in \mathbb{C} : |z| = 2\}$, so the length of the contour γ is $\frac{2\pi(2)}{4} = \pi$. Therefore, we have

$$\left| \int_\gamma \frac{dz}{z^2 - 1} \right| \leq \int_\gamma \frac{dz}{3} = \frac{\pi}{3}.$$

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6. Let γ_R be the arc of the circle $\{z \in \mathbb{C} : |z| = R\}$ from $z = R$ to $z = -R$ that lies in the upper half plane, where $R > 1$. Show that
$$\left| \int_{\gamma_R} \frac{z^2}{z^6 + 1} dz \right| \leq \frac{\pi R^3}{R^6 - 1},$$
and hence show that
$$\lim_{R \to +\infty} \int_{\gamma_R} \frac{z^2}{z^6 + 1} dz = 0.$$

Solution. For any $z \in \gamma_R$, we have $|z^6 + 1| \geq |z|^6 - 1 = R^6 - 1 > 0$. Moreover, the length of the contour γ_R is $\frac{2\pi R}{2} = \pi R$. Therefore, we have

$$\begin{aligned}
\left| \int_{\gamma_R} \frac{z^2}{z^6 + 1} dz \right| &\leq \int_{\gamma_R} \frac{|z|^2}{|z^6 + 1|} dz \\
&\leq \int_{\gamma_R} \frac{R^2}{R^6 - 1} dz \\
&= \frac{\pi R^3}{R^6 - 1}
\end{aligned}$$

As $R \to \infty$, it is easy to see that $\frac{\pi R^3}{R^6 - 1} \to 0$, hence

$$\lim_{R \to +\infty} \int_{\gamma_R} \frac{z^2}{z^6 + 1} dz = 0.$$

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Optional Part

1. Find the domain over which the function

$$f(z) = f(x+iy) = \left|x^2 - y^2\right| + 2i\,|xy|$$

is analytic.

Solution. Let $u(x,y) = |x^2 - y^2|$ and $v(x,y) = 2\,|xy|$. If $(x_0, y_0) \in \mathbb{R}^2$ satisfying $u(x_0,y_0) \neq 0$ and $v(x_0,y_0) \neq 0$, we can compute their parital derivatives:

$$u_x(x_0,y_0) = 2x_0 \frac{x_0^2 - y_0^2}{|x_0^2 - y_0^2|} \qquad u_y(x_0,y_0) = -2y_0 \frac{x_0^2 - y_0^2}{|x_0^2 - y_0^2|}$$

$$v_x(x_0,y_0) = 2y_0 \frac{x_0 y_0}{|x_0 y_0|} \qquad v_y = 2x_0 \frac{x_0 y_0}{|x_0 y_0|}$$

We observe that the Cauchy-Riemann equations hold if and only if $\frac{x_0^2-y_0^2}{|x_0^2-y_0^2|} = \frac{x_0y_0}{|x_0y_0|}$. That is, x_0y_0 and $x_0^2 - y_0^2$ have the same sign. The complex plane $\mathbb{C}$ is partitioned into 8 regions by 4 straight lines, namely $\{x=0\}$, $\{y=0\}$, $\{x=y\}$ and $\{x=-y\}$. In the polar coordinate, the 8 regions are respectively $\{0<\theta<\pi/4\}$, $\{\pi/4<\theta<\pi/2\}$, $\{\pi/2<\theta<3\pi/4\}$, $\{3\pi/4<\theta<\pi\}$, $\{\pi<\theta<5\pi/4\}$, $\{5\pi/4<\theta<3\pi/2\}$, $\{3\pi/2<\theta<7\pi/4\}$ and $\{7\pi/4<\theta<2\pi\}$. In order for xy and x^2-y^2 to have the same sign, (x,y) must lie in the regions $\{0<\theta<\pi/4\}$, $\{\pi/2<\theta<3\pi/4\}$, $\{\pi<\theta<5\pi/4\}$ and $\{3\pi/2<\theta<7\pi/4\}$. Moreover, for any point outside these regions, its neighborhood must intersect one of the other 4 regions, i.e. $\{\pi/4<\theta<\pi/2\}$, $\{3\pi/4<\theta<\pi\}$, $\{5\pi/4<\theta<3\pi/2\}$ and $\{7\pi/4<\theta<2\pi\}$, where f is not differentiable.

Therefore, the domains over which f is analytic, are $\{0<\theta<\pi/4\}$, $\{\pi/2<\theta<3\pi/4\}$, $\{\pi<\theta<5\pi/4\}$ or $\{3\pi/2<\theta<7\pi/4\}$. ◀

2. Suppose that $f(z)$ is analytic on a domain D, where D is symmetric with respect to the real axis. Show that $g(z) := \overline{f(\bar{z})}$ is a well-defined analytic function on D.

Solution. Let u, v be the real-valued functions on D such that $f(x+iy) = u(x,y) + iv(x,y)$. Since D is symmetric with respect to the real axis, $u(x,y)$ is well-defined if and only if $u(x,-y)$ is well-defined. This is also true for the function $v(x,y)$. Note that

$$g(x+iy) = \overline{f(x-iy)} = u(x,-y) - iv(x,-y).$$

If we put $p(x,y), q(x,y)$ be the real part and imaginary part of g, then their partial derivatives at (x_0,y_0) are given by:

$$p_x(x_0,y_0) = u_x(x_0,-y_0) \qquad p_y(x_0,y_0) = -u_y(x_0,-y_0)$$
$$q_x(x_0,y_0) = -v_x(x_0,-y_0) \qquad q_y(x_0,y_0) = v_y(x_0,-y_0)$$

Since $u_x = v_y$ and $u_y = -v_x$, it follows that $p_x = q_y$ and $p_y = -q_x$. Moreover, the function $(x,y) \mapsto (p(x,y), q(x,y))$ is just the composite function

$$(x,y) \mapsto (x,-y) \mapsto (u(x,-y), v(x,-y)) \mapsto (u(x,-y), -v(x,-y)),$$

hence it is differentiable. Therefore, g is complex differentiable at every point of D. (see Week 2 Lecture notes). ◀

3. Let γ_R be the circle $\{z \in \mathbb{C} : |z| = R\}$ in the counterclockwise direction. Show that, for $R > 2$,
$$\left| \int_{\gamma_R} \frac{3z-1}{z^4+4z^2+3} dz \right| \leq \frac{2\pi R(3R+1)}{(R^2-1)(R^2-3)}.$$

Solution. On the circle $\{z \in \mathbb{C} : |z| = R\}$, $|3z-1| \leq 3|z|+1 = 3R+1$, and $|z^4+4z^2+3| = |(z^2+1)(z^2+3)| \geq (|z|^2-1)(|z|^2-3) = (R^2-1)(R^2-3)$. Therefore, we have
$$\left| \frac{3z-1}{z^4+4z^2+3} \right| \leq \frac{3R+1}{(R^2-1)(R^2-3)} \qquad \text{for every } |z| = R.$$
Since the length of the contour is $2\pi R$, the result follows. ◀

4. Let γ_R be the vertical line segment from R to $R+4\pi i$, where $R > 0$. Show that
$$\left| \int_{\gamma_R} \frac{2e^z}{1+e^{3z}} dz \right| \leq \frac{8\pi e^R}{e^{3R}-1}.$$

Solution. For every $z \in \gamma_R$, $z = R+iy$ for some $0 \leq y \leq 4\pi$, hence we have $|2e^z| = 2e^R$ and $|1+e^{3z}| \geq |e^{3z}| - 1 = e^{3R} - 1$. Since the length of the contour is 4π, the result follows. ◀

5. Does the function $f(z) = \dfrac{1}{z^2}$ defined on $\mathbb{C} \setminus \{0\}$ have an antiderivative?

Solution. Yes, $-\frac{1}{z}$ is an antiderivative for the function $f(z)$. However, the function $\frac{1}{z}$ has no antiderivative on the domain $\mathbb{C} \setminus \{0\}$. This can be checked from the calculation that
$$\int_{|z|=1} \frac{dz}{z} = \int_0^{2\pi} \frac{1}{e^{it}} ie^{it} dt = 2\pi i \neq 0.$$

You can also argue in this way: since the function $f(z)$ is analytic in $D := \mathbb{C} \setminus \{0\}$, to claim that $f(z)$ has an antiderivative, it suffices to check that
$$\int_{|z|=1} f(z) dz = 0.$$

In general, you need to check that $\int_\gamma f(z)dz = 0$ for every closed contour γ in D, but by the analyticity of f and the Cauchy-Goursat theorem, you only need to evaulate that particular contour. ◀