THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MMAT5220 Complex Analysis and Its Applications 2019-20
Homework 1
Due Date: 20th February 2020

Compulsory Part

1. Let $n \geq 1$.
   (a) Show that $1 + z + z^2 + \cdots + z^n = \frac{1-z^{n+1}}{1-z}$ if $z \neq 1$.
   (b) Use part (a) to deduce **Lagrange's trigonometric identity**:
   $$1 + \cos\theta + \cos 2\theta + \cdots + \cos n\theta = \frac{1}{2} + \frac{\sin(n+\frac{1}{2})\theta}{2\sin\frac{\theta}{2}}$$
   when θ is not a multiple of 2π.

2. Show that $|z_1 - z_2| \geq ||z_1| - |z_2||$ for any $z_1, z_2 \in \mathbb{C}$.

3. Consider the function
   $$T(z) = \frac{az+b}{cz+d},$$
   where $ad - bc \neq 0$. Show that
   (a) $\lim_{z\to\infty} T(z) = \infty$ if $c = 0$;
   (b) $\lim_{z\to\infty} T(z) = \frac{a}{c}$ and $\lim_{z\to -d/c} T(z) = \infty$ if $c \neq 0$.

4. For the following functions defined on the whole complex plane, show that they are complex differentiable at every point by computing the partial derivatives of their real and imaginary parts and verifying the Cauchy-Riemann equations:
   (a) $f(z) = z^2$.
   (b) $f(z) = e^z$.

   (Remark: Functions which are complex differentiable on the whole complex plane are called **entire functions**.)

5. Consider the function $f : \mathbb{C} \to \mathbb{C}$ defined by $f(z) = \bar{z}$. By considering the Cauchy-Riemann equations, show that $f'(z)$ does not exist at any point.

Optional Part

1. Show, by definition, that
   (a) $e^{z_1+z_2} = e^{z_1} \cdot e^{z_2}$ for any $z_1, z_2 \in \mathbb{C}$;
   (b) $\log(z_1 z_2) = \log z_1 + \log z_2$ for any $z_1, z_2 \in \mathbb{C} \setminus \{0\}$.
   (c) $\sin^2 z + \cos^2 z = 1$ for any $z \in \mathbb{C}$.

2. Suppose $\lim_{z\to z_0} f(z) = 0$ and there exists a positive real number M such that $|g(z)| \leq M$ for all z in some neighborhood of z_0. Show that $\lim_{z\to z_0} f(z)g(z) = 0$.

3. Show that the following are entire functions by computing the partial derivatives of their real and imaginary parts and verifying the Cauchy-Riemann equations:

   (a) $f(z) = \sin z$.

   (b) $f(z) = \cos z$.

   (c) $f(z) = \sinh z := \frac{e^z - e^{-z}}{2}$.

   (d) $f(z) = \cosh z := \frac{e^z + e^{-z}}{2}$.

4. Let f be a function on a domain $D \subset \mathbb{C}$ such that both f and $\bar{f}$ are analytic. Show that f must be a constant function.