MATH 5010: Linear Analysis: Complementary Exercise
December 2019

1. (i) Let X be a normed space. Suppose that every 2-dimensional subspace of X is an inner product space. Show that X is an inner product space.

   Proof:: Recall that a normed space is an inner product space if and only if for each pair of vectors x and y in X must satisfy the Parallelogram identity, i.e., $\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$. Part (i) follows from this fact directly.

   (ii) Let (x_n) and (y_n) be the sequences in a Hilbert space H. Suppose that the limits $\lim \|x_n\|, \lim \|y_n\|$ and $\lim \|\frac{x_n+y_n}{2}\|$ exist and are equal. Show that if (x_n) is convergent, then so is (y_n).

   Proof: Notice that Parallelogram identity gives

$$\|\frac{x_n+y_n}{2}\|^2 + \|\frac{x_n-y_n}{2}\|^2 = \frac{1}{2}(\|x_n\|^2 + \|y_n\|^2)$$

for all n. Thus, if we put $L := \lim \|x_n\| = \lim \|y_n\| = \lim \|\frac{x_n+y_n}{2}\|$, then we see that

$$\|\frac{x_n-y_n}{2}\|^2 = \frac{1}{2}(\|x_n\|^2+\|y_n\|^2) - \|\frac{x_n+y_n}{2}\|^2 \longrightarrow \frac{1}{2}(L^2+L^2) - L^2 = 0 \quad \text{as } n \to \infty.$$

This implies that $\lim \|x_n - y_n\| = 0$ and thus, we have $\lim y_n = \lim x_n + \lim(y_n - x_n)$. The proof is finished.

2. Let $D := \{x \in \ell^2 : \sum_{n=1}^{\infty} n^2 |x(n)|^2 < \infty\}$. Define a linear operator $T : D \to \ell^2$ by $Tx(n) := nx(n)$ for $x \in D$ and $n = 1, 2....$ Show that the operator T satisfies the condition $(Tx, y) = (x, Ty)$ for all $x, y \in D$ but T is not bounded.

   Proof: We first show that we have $(Tx, y) = (y, Tx)$ for all $x, y \in D$. In fact, we have

$$(Tx, y) = \sum_{n=1}^{\infty} (Tx(n))\overline{y(n)} = \sum_{n=1}^{\infty} nx(n)\overline{y(n)} = (x, Ty).$$

   Next, we claim that T is unbounded. In fact, we let $e_k(n) = 1$ if $n = k$, otherwise, is equal to 0. Then we have $e_k \in D$ and $\|e_k\| =$. Notice that we have $\|Te_k\| = k$ for all $k = 1, 2, ...$ Thus, the map T is unbounded. The proof is finished.

3. For each $x \in \ell^\infty$, define a linear operator M_x from ℓ^2 to itself by $M_x(\xi)(k) := x(k)\xi(k)$ for $\xi \in \ell^2$ and $k = 1, 2.....$

(i) Show that $\|M_x\| = \|x\|_\infty$ for any $x \in \ell^\infty$.

Proof Notice that for each $\xi \in \ell^2$, we have

$$\|M_x(\xi)\| = \sum_{k=1}^{\infty} |x(k)\xi(k)|^2 \leq \|x\|_\infty \sum_{k=1}^{\infty} |\xi(k)|^2 = \|x\|_\infty \|\xi\|_2.$$

This gives $\|M_x\| \leq \|x\|_\infty$. It remains to show that $\|M_x\| \geq \|x\|_\infty$. Fix a positive integer k and let $e_k \in \ell^2$ be as in Question 2. Then we have $|x(k)| = |M_x(e_k)| \leq \|M_x\|$ as desired.

(ii) Show that M_x is selfadjoint if and only if $x = \bar{x}$, where $\bar{x}(k) := \overline{x(k)}$. (See Prop 10.3)

Proof: Recall that an operator T from a Hilbert space H to itself is said to be selfadjoint if $(Tu, v) = (u, Tv)$ for all $u, v \in H$.
Now if $x = \bar{x}$, then we have

$$(M_x u, v) = \sum_{k=1}^{\infty} x(k)u(k)\overline{v(k)} = \sum_{k=1}^{\infty} u(k)\overline{x(k)v(k)} = (u, M_x v)$$

for all $u, v \in H$. Thus, M_x is selfadjoint.
Conversely, if M_x is selfadjoint, then we consider $e_k \in \ell^2$ as above, we have $(M_x e_k, e_k) = x(k)$ and $(e_k, M_x e_k) = \overline{x(k)}$ for all $k = 1, 2, ...$ The proof is finished.

4. Let X be a normed space and let S_{X^*} be the closed unit sphere of X^*. Suppose that there is $0 < r < 1$ such that $S_{X^*} \subseteq \bigcup_{k=1}^{n} B(x_k^*, r)$ for some $x_1^*, ..., x_n^*$ in X^* with $\|x_k^*\| = 1$ for all $k = 1, ..., n$.
Define a linear map $T : X \to c_0$ by

$$T(x) = (x_1^*(x), ..., x_n^*(x), 0, ...) \in c_0.$$

(i) Show that $\|T\| = 1$.

Proof If $x \in X$ with $\|x\| \le 1$, then we have $|x_k^*(x)| \le 1$ for all $k = 1, 2, ..n$ and hence, we have $\|T\| \le 1$. On the other hand, since $|M_x(e_k)| = 1$, we have $\|T\| = 1$ as desired.

(ii) Show that $\|x\| \le \frac{1}{1-r}\|Tx\|$ for all $x \in X$ (Hint: Use Prop 5.9).

Proof Notice that there is an element $f \in X^*$ with $\|f\| = 1$ such that $\|x\| = f(x)$. Also, by the assumption, there is x_k^* such that $\|x_k^* - f\| < r$. This implies that

$$\|x\| = |f(x)| \le |((f - x_k^*)(x)| + |x_k^*(x)| \le r\|x\| + \|Tx\|.$$

This gives $\|x\| \le \frac{1}{1-r}\|Tx\|$. The proof is finished.

(iii) Show that the operator is an isomorphism from X onto a subspace $T(X)$ of c_0 and $\|T^{-1}\| \le 1/(1-r)$.

Proof Note that Part (ii) implies that the map T is injective and

$$\|T^{-1}(Tx)\| = \|x\| \le \frac{1}{1-r}\|Tx\|$$

for all $x \in X$ and hence, we have $\|T^{-1}\| \le 1/(1-r)$.

End