Math4230 Tutorial 9

1. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function. Suppose $x^*$ is a local minimizer of $f$, show that it is also a global minimizer.

2. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a strictly convex function. Suppose $f$ has a global minimizer, show that it is unique.

3. Let $f : Y \to \mathbb{R}$ be a Lipschitz continuous function with constant $L$. Let $X$ be a nonempty closed subset of $Y$, and $c$ be a number such that $c > L$.
   (a) Show that if $x^*$ minimizes $f$ over $X$, then $x^*$ minimizes
       $$f_c(x) = f(x) + c \inf_{\tau \in X} ||\tau - x||$$
       over $Y$.
   (b) Show that if $x^*$ minimizes $f_c(x)$ over $Y$, then $x^* \in X$, and hence $x^*$ minimizes $f$ over $X$.

4. Consider the following problem
   $$\min x^2 + 1 \text{ subject to } (x - 2)(x - 4) \leq 0$$
   (a) Find the feasible set, optimal value and the optimal solution.
   (b) Write down the Lagrangian $L(x, \lambda)$. Find the dual function $q$.
   (c) Solve the dual problem. Does strong duality hold?

5. Consider the following problem
   $$\min \langle c, x \rangle, \text{ subject to } f(x) \leq 0$$
   with $c \neq 0$.
   Express the dual problem in terms of the conjugate function of $f$. 