Math4230 Tutorial 6 Solution

1. (a) \( x - \frac{\langle a, x \rangle - b}{\|a\|} a \)

   (b) \( \max\{x, 0\} \)

2. 
   \[
   \|P_C(x_1) - P_C(x_2)\|^2
   = (P_C(x_1) - x_2, P_C(x_1) - P_C(x_2)) + \langle x_2 - P_C(x_2), P_C(x_1) - P_C(x_2) \rangle
   \leq (P_C(x_1) - x_2, P_C(x_1) - P_C(x_2))
   = (P_C(x_1) - x_1, P_C(x_1) - P_C(x_2)) + \langle x_1 - x_2, P_C(x_1) - P_C(x_2) \rangle
   \leq (x_1 - x_2, P_C(x_1) - P_C(x_2))
   \]

3. Since \( C \) is closed and \( \bar{x} \notin C \), \( C \) and \( \bar{x} \) can be strictly separated by a hyperplane. Hence for some nonzero \( a \in \mathbb{R}^n \) we have,

   \( \langle a, x \rangle < \langle a, \bar{x} \rangle \), \( \forall x \in C \).

   Suppose \( \langle a, x' \rangle > 0 \) for some \( x' \in C \). Since \( C \) is a cone, \( \lambda x \in C \) \( \forall \lambda > 0 \).
   By choosing a large \( \lambda \), we get a contradiction, since \( \langle a, \lambda x' \rangle > \langle a, \bar{x} \rangle \).
   Hence \( \langle a, x \rangle \leq 0 \), \( \forall x \in C \).
   Since \( C \) is a closed cone, \( 0 \in C \). We must have \( \langle a, \bar{x} \rangle > 0 \).

4. Since \( V \) is closed and \( \bar{x} \notin V \), \( V \) and \( \bar{x} \) can be strictly separated by a hyperplane. Hence for some nonzero \( a \in \mathbb{R}^n \) we have,

   \( \langle a, x \rangle < \langle a, \bar{x} \rangle \), \( \forall x \in V \).

   Suppose \( \langle a, x' \rangle \neq 0 \) for some \( x' \in V \). Since \( V \) is a subspace, \( \lambda x \in V \) for all \( \lambda \). We can choose \( \lambda \) such that \( \langle a, \lambda x' \rangle > \langle a, \bar{x} \rangle \). Hence, we get a contradiction. Since \( V \) is a subspace, \( 0 \in V \). We must have \( \langle a, \bar{x} \rangle > 0 \).