1. \( \overline{\text{ri}(C)} \subset C \) since \( \text{ri}(C) \subset C \).

Conversely, suppose \( x \in C \).

Let \( x \in \text{ri}(C) \). Consider \( x_k = \frac{1}{k}x + (1 - \frac{1}{k})x \).

By the line segment property, each \( x_k \in \text{ri}(C) \). Also, \( x_k \to x \). Therefore, \( x \in \text{ri}(C) \).

2. We first prove that \( \text{ri}(C) = \text{ri}(\overline{C}) \). \( \text{ri}(C) \subset \text{ri}(\overline{C}) \) follows from the definition and the fact that \( \text{aff}(C) = \text{aff}(\overline{C}) \).(Try to show this)

Conversely, suppose \( x \in \text{ri}(\overline{C}) \). Suppose \( x \in \text{ri}(C) \). (which exists since \( \text{ri}(C) \) is nonempty)

We may assume \( x \neq z \). Then by Prolongation lemma, \( y = x + \gamma(x - z) \in \overline{C} \), for some \( \gamma > 0 \).

Then \( x = \frac{\gamma}{1+\gamma}x + \frac{1}{1+\gamma}y \). By Line Segment Property, \( x \in \text{ri}(C) \).

Now, since \( C_1 = C_2 \), \( \text{ri}(C_1) = \text{ri}(C_2) \). Hence, \( \text{ri}(C_1) = \text{ri}(C_2) \).

3. (a) \( C_1 = \{(x, y) \mid 0 \leq x \leq 1, \ y = 0\} \)

\( C_2 = \{(x, y) \mid 0 \leq x \leq 1, \ 0 \leq y \leq 1\} \)

(b) Let \( x \in \text{ri}(C_1) \). Then there exists \( \epsilon > 0 \) such that \( B(x, \epsilon) \cap \text{aff}(C_1) \subset C_1 \).

But \( \text{aff}(C_1) = \text{aff}(C_2) \). So \( B(x, \epsilon) \cap \text{aff}(C_2) \subset C_1 \subset C_2 \).

Hence \( x \in \text{ri}(C_2) \).

4. Let \( x^* \in X^* \cap \text{ri}(X) \). Let \( x \in X \).

By Prolongation lemma, \( y = x^* + \gamma(x^* - x) \in X \).

So \( x^* = \frac{\gamma}{1+\gamma}x + \frac{1}{1+\gamma}y \). Since \( f \) is concave, we have

\[
f(x^*) \geq \frac{\gamma}{1+\gamma}f(x) + \frac{1}{1+\gamma}f(y) \geq \frac{\gamma}{1+\gamma}f(x^*) + \frac{1}{1+\gamma}f(x^*) = f(x^*)
\]

since \( f(x) \geq f(x^*) \), \( f(y) \geq f(x^*) \).

So we must have equality. In particular, \( f(x) = f(x^*) \). This holds for any \( x \in X \). Hence, \( f \) must be constant.