

Math4230 Tutorial 3 Solution

1. Let $x, y \in C$, and let $\alpha \in [0, 1]$. Then

$$\begin{aligned}
 h(\alpha x + (1 - \alpha)y) &= g(f(\alpha x + (1 - \alpha)y)) \\
 &= g(f_1(\alpha x + (1 - \alpha)y), \dots, f_m(\alpha x + (1 - \alpha)y)) \\
 &\leq g(\alpha f_1(x) + (1 - \alpha)f_1(y), \dots, \alpha f_m(x) + (1 - \alpha)f_m(y)) \\
 &= g(\alpha(f_1(x), \dots, f_m(x)) + (1 - \alpha)(f_1(y), \dots, f_m(y))) \\
 &\leq \alpha g(f_1(x), \dots, f_m(x)) + (1 - \alpha)g(f_1(y), \dots, f_m(y)) \\
 &= \alpha g(f(x)) + (1 - \alpha)g(f(y)) \\
 &= \alpha h(x) + (1 - \alpha)h(y)
 \end{aligned}$$

2. (a) $z' \nabla^2 f_1(x) z = \frac{1}{(e^{x_1} + \dots + e^{x_n})^2} \sum_{i=1}^n \sum_{j=1}^n e^{x_i + x_j} (z_i - z_j)^2 \geq 0, \forall z \in \mathbb{R}^n$.
Hence the Hessian of f_1 is positive semidefinite at all $x \in \mathbb{R}^n$ and f_1 is convex.

(b) Let $f(x) = \|x\|^p, g(t) = t^p$. Then use the result of (1).

(c) Let $f(x) = x'Ax, g(t) = e^t$. Then use the result of (1).

3. a) We show that f is strongly convex iff

$$f(y) \geq f(x) + \langle \nabla f(x), y - x \rangle + \frac{\alpha}{2} \|y - x\|^2, \forall x, y$$

Suppose f is strongly convex. Consider $g(t) = f(x + t(y - x))$. Then $g'(t) = \langle \nabla f(x + t(y - x)), (y - x) \rangle$.

Then

$$\begin{aligned}
 f(y) - f(x) - \langle \nabla f(x), y - x \rangle &= g(1) - g(0) - \langle \nabla f(x), y - x \rangle \\
 &= \int_0^1 \langle \nabla f(x + t(y - x)), y - x \rangle - \langle \nabla f(x), y - x \rangle dt \\
 &= \int_0^1 \langle \nabla f(x + t(y - x)) - \nabla f(x), y - x \rangle dt \\
 &\geq \int_0^1 \alpha t \|y - x\|^2 dt \\
 &= \frac{\alpha}{2} \|y - x\|^2
 \end{aligned}$$

The other direction is simple.

For $x \neq y, \|y - x\| \neq 0$. Hence, the above shows that f is strictly convex if f is strongly convex.

b) Consider $g(x) = f(x) - \frac{\alpha}{2} \|x\|^2$. $\nabla g(x) = \nabla f(x) - \alpha x$.
 g is convex iff

$$g(y) \geq g(x) + \langle \nabla g(x), y - x \rangle, \forall x, y$$

This is also equivalent to

$$f(y) \geq f(x) + \langle \nabla f(x), y - x \rangle + \frac{\alpha}{2} \|y - x\|^2, \forall x, y$$

Hence f is strongly convex iff g is convex.

Note that $\nabla^2 g(x) = \nabla^2 f(x) - \alpha I$.

This shows that f is strongly convex iff $\nabla f(x) - \alpha I$ is positive semi-definite.

4. Suppose f is positively homogeneous and convex. Then

$$f(x + y) = 2f\left(\frac{1}{2}x + \frac{1}{2}y\right) \leq 2\left(\frac{1}{2}f(x) + \frac{1}{2}f(y)\right) = f(x) + f(y)$$

Hence f is subadditive.

Suppose f is positively homogeneous and subadditive. Let $\lambda \in [0, 1]$.

$$f(\lambda x + (1 - \lambda)y) \leq f(\lambda x) + f((1 - \lambda)y) = \lambda f(x) + (1 - \lambda)f(y)$$

Hence, f is convex.