1 Convex Sets and Functions

1.1 Convex Sets

Definition: (Convex sets) A subset \( C \) of \( \mathbb{R}^n \) is called convex if
\[
\lambda x + (1 - \lambda)y \in C, \quad \forall \ x, y \in C, \ \forall \lambda \in [0,1].
\]

Geometrically, it just means that the line segment joining any two points in a convex set \( C \) lies in \( C \).

![Figure 1: Examples of convex and non-convex set](image)

Definition: (Convex combination) Given \( x_1, \ldots, x_m \in \mathbb{R}^n \), an element in the form \( x = \sum_{i=1}^{m} \lambda_i x_i \), where \( \sum_{i=1}^{m} \lambda_i = 1 \) and \( \lambda_i \geq 0 \) is called a convex combination of \( x_1, \ldots, x_m \).

Proposition: A subset \( C \) of \( \mathbb{R}^n \) is convex if and only if contains all convex combination of its element.

Proof. Suppose \( C \) is convex. We will show by induction that it contains all convex combination \( \sum_{i=1}^{m} \lambda_i x_i \) of its elements.

The case \( m = 1, 2 \) is trivial, so suppose all convex combination of \( k \) elements lies in \( C \), where \( k \leq m \). Consider
\[
x := \sum_{i=1}^{m+1} \lambda_i x_i, \quad \text{where} \quad \sum_{i=1}^{m+1} \lambda_i = 1
\]

If \( \lambda_{m+1} = 1 \), then \( \lambda_1 = \cdots = \lambda_m = 0 \). Then \( x \in C \). So assume \( \lambda_{m+1} < 1 \), then
\[
\sum_{i=1}^{m} \lambda_i = 1 - \lambda_{m+1} \quad \text{and} \quad \sum_{i=1}^{m} \frac{\lambda_i}{1 - \lambda_{m+1}} = 1
\]
Then \( y = \sum_{i=1}^m \frac{\lambda_i}{1-\lambda_{m+1}} x_i \in C \). Hence
\[
x = (1 - \lambda_{m+1})y + \lambda_{m+1}x_{m+1} \in C
\]
The other direction is trivial.

**Proposition:** Let \( C_1 \) be a convex set of \( \mathbb{R}^n \) and let \( C_2 \) be a convex set pf \( \mathbb{R}^m \). Then the Cartesian product \( C_1 \times C_2 \) is a convex subset of \( \mathbb{R}^n \times \mathbb{R}^m \).

1.1.1 **Examples of Convex Sets**

(a) Open and closed balls in \( \mathbb{R}^n \).
(b) **Hyperplanes:** \( \{ x : \langle a, x \rangle = b, a \in \mathbb{R}^n, b \in \mathbb{R} \} \).
(c) **Halfspaces:** \( \{ x : \langle a, x \rangle \leq b, a \in \mathbb{R}^n, b \in \mathbb{R} \} \).
(d) **Non-Negative Orthant:** \( \mathbb{R}^n_+ = \{ x \in \mathbb{R}^n : x \geq 0 \} \).
(e) **Convex cones:** A set \( C \) is called a cone if \( \alpha x \in C, \forall \alpha > 0, x \in C \). A set which is convex is called a convex cone.

**Proposition:** Let \( \{ C_i \mid i \in I \} \) be a collection of convex sets. Then:

(a) \( \bigcap_{i \in I} C_i \) is convex, where each \( C_i \) is convex.
(b) \( C_1 + C_2 = \{ x + y : x \in C_1, y \in C_2 \} \) is convex.
(c) \( \lambda C \) is convex for any convex sets \( C \) and scalar \( \lambda \). Furthermore, \( (\lambda_1 + \lambda_2)C = \lambda_1 C + \lambda_2 C \) for positive \( \lambda_1, \lambda_2 \).
(d) \( C^o, \overline{C} \) are convex, i.e. the interior and closure of a convex set are convex.
(e) \( T(C), T^{-1}(C) \) are convex, where \( T \) is a linear map.

**Proof.** Parts (a)-(c), (e) follows from the definition (Exercise!). Let’s prove (d).

**Interior** Let \( x, y \in C^o \). Then there exists \( r \) such that balls with radius \( r \) centred at \( x \) and \( y \) are both inside \( C \).

Suppose \( \lambda \in [0,1] \) and \( ||z|| < r \). By convexity of \( C \), we have,
\[
\lambda x + (1 - \lambda) y + z = \lambda(x + z) + (1 - \lambda)(y + z) \in C
\]
Therefore, \( \lambda x + (1 - \lambda)y \in C^o \). Hence \( C^o \) is convex.

**Closure** Let \( x, y \in \overline{C} \). Then there exists sequences \( \{x_k\} \subseteq C, \{y_k\} \subseteq C \) such that \( x_k \to x, y_k \to y \). Suppose \( \alpha \in [0,1] \). Then for each \( k \),
\[
\alpha x_k + (1 - \alpha) y_k \in C
\]
But \( \alpha x_k + (1 - \alpha)y_k \to \lambda x + (1 - \lambda)y \in \overline{C} \). Hence, \( \overline{C} \) is convex. \( \square \)
1.2 Convex and Affine Hulls

1.2.1 Convex Hull

Definition: (Convex Hull)
Let \( X \) be a subset of \( \mathbb{R}^n \). The convex hull of \( X \) is defined by
\[
\text{conv}(X) := \bigcap \{ C | C \text{ is convex and } X \subseteq C \}
\]
In other words, \( \text{conv}(X) \) is the smallest convex set containing \( X \).

The next proposition provides a good representation for elements in the convex hull.

**Proposition:** For any subset \( X \) of \( \mathbb{R}^n \),
\[
\text{conv}(X) = \left\{ \sum_{i=1}^{m} \lambda_i x_i \mid \sum_{i=1}^{m} \lambda_i = 1, \lambda_i \geq 0, x_i \in X \right\}
\]

**Proof.** Let \( C = \left\{ \sum_{i=1}^{m} \lambda_i x_i \mid \sum_{i=1}^{m} \lambda_i = 1, \lambda_i \geq 0, x_i \in X \right\} \). Clearly, \( X \subseteq C \).
Next, we check that \( C \) is convex.
Let \( a = \sum_{i=1}^{p} \alpha_i a_i, b = \sum_{j=1}^{q} \beta_j b_j \) be elements of \( C \), where \( a_i, b_i \in C \) with \( \alpha_i, \beta_j \geq 0 \) and \( \sum \alpha_i = \sum \beta_j = 1 \). Suppose \( \lambda \in [0,1] \), then
\[
\lambda a + (1 - \lambda)b = \sum_{i=1}^{p} \lambda \alpha_i a_i + \sum_{j=1}^{q} (1 - \lambda) \beta_j b_j.
\]
Since
\[
\sum_{i=1}^{p} \lambda \alpha_i + \sum_{j=1}^{q} (1 - \lambda) \beta_j = \lambda \sum_{i=1}^{p} \alpha_i + (1 - \lambda) \sum_{j=1}^{q} \beta_j = 1
\]
we have \( \lambda a + (1 - \lambda)b \in C \). Hence, \( C \) is convex. Also, \( \text{conv}(X) \subseteq C \) by the definition of \( \text{conv}(X) \).
Suppose \( a = \sum \lambda_i a_i \in C \). Then since each \( a_i \in X \subseteq \text{conv}(X) \) and \( \text{conv}(X) \) is convex, we have \( a \in \text{conv}(X) \). Therefore, \( \text{conv}(X) = C \). \( \square \)