MATH 4050 Real Analysis

Suggested Solution of Homework 8

Only the solutions to $*$ questions are provided.

1.* (3rd: P.89, Q9; 4th: P.84, Q22)
Let $\{f_n\}$ be a sequence of nonnegative measurable functions on $(-\infty, +\infty)$ such that $f_n \to f$ a.e., and suppose $\displaystyle\int f_n \to \int f < \infty$. Show that for each measurable set E we have $\displaystyle\int_E f_n \to \int_E f$.

Solution. Applying Fatou's Lemma to the sequence $\{f_n\}$ on E, we have

$$\int_E f \leq \liminf \int_E f_n.$$

As $\{f_n - f_n\chi_E\}$ is a sequence of nonnegative measurable functions that converges to $f - f\chi_E$ a.e., it follows from Fatou's Lemma that

$$\int (f - f\chi_E) \leq \liminf \int (f_n - f_n\chi_E)$$

Since $\displaystyle\int f < \infty$ and $\displaystyle\int f_n \to \int f$, we have $\displaystyle\int f_n < \infty$ for all large n. The above inequality becomes

$$\int f - \int_E f \leq \lim \int f_n + \liminf \left(-\int_E f_n \right) = \int f - \limsup \int_E f_n.$$

Thus $\displaystyle\limsup \int_E f_n \leq \int_E f$. Therefore $\displaystyle\lim \int_E f_n = \int_E f$. ◀

2.* (3rd: P.93, Q10)

(a) Show that if f is integrable over E, then so is $|f|$ and

$$\left| \int_E f \right| \leq \int_E |f|.$$

Does the integrability of $|f|$ imply that of f?

(b) The imporoper Riemann integral of a function may exist without the function being integrable (in the sense of Lebesgue), e.g., if $f(x) = \dfrac{\sin x}{x}$ on $[0, \infty]$. If f is integrable, show that the improper Riemann integral is equal to the Lebesgue integral when the former exists.

Solution. (a) Suppose f is integrable over E. Then f^+ and f^- are both integrable over E. So $|f| = f^+ + f^-$ is also integrable over E. Moreover,

$$\left|\int f\right| = \left|\int f^+ - \int f^-\right| \leq \left|\int f^+\right| + \left|\int f^-\right| = \int f^+ + \int f^- = \int |f|.$$

The converse is true if the measurability of f is assumed. If $|f|$ is integrable over E, then both f^+, f^- are integrable over E since $f^+ \leq |f|$ and $f^- \leq |f|$. Thus f is integrable over E.

(b) Without loss of generality, we only consider the improper Riemann integral of the form $(\mathcal{R})\displaystyle\int_a^b f = \lim_{c\to b^-}(\mathcal{R})\int_a^c f$. Suppose f is Lebesgue integrable and the improper Riemann integral exists. In particular, f is Riemann integrable on any $[a,c] \subseteq [a,b)$. Let $\{c_n\}$ be a sequence of real numbers in (a,b) that increases to b. Let $f_n = f\chi_{[a,c_n]}$. Then $f_n \to f$ on $[a,b)$ and $|f_n| \leq |f|$. So, by Dominated Convergence Theorem,

$$(\mathcal{R})\int_a^b f = \lim_n (\mathcal{R})\int_a^{c_n} f = \lim_n \int_a^{c_n} f = \lim_n \int f_n = \int_a^b f.$$

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4.* (3rd: P.93, Q12; 4th: P.89, Q30)
Let g be an integrable function on a set E and suppose that $\{f_n\}$ is a sequence of measurable functions such that $|f_n(x)| \leq g(x)$ a.e. on E. Show that

$$\int_E \liminf f_n \leq \liminf \int_E f_n \leq \limsup \int_E f_n \leq \int_E \limsup f_n.$$

Solution. Note that $\{g + f_n\}$ is a sequence of nonnegative measurable functions on E. By (Generalized) Fatou's Lemma,

$$\int_E g + \int_E \underline{\lim}\, f_n = \int_E \underline{\lim}(g + f_n) \leq \underline{\lim} \int_E (g + f_n) = \int_E g + \underline{\lim} \int_E f_n.$$

Thus $\displaystyle\int_E \underline{\lim}\, f_n \leq \underline{\lim} \int_E f_n$. Similarly $\{g - f_n\}$ is a sequence of nonnegative measurable functions on E. It follows from (Generalized) Fatou's Lemma that

$$\int_E g + \int_E \underline{\lim}(-f_n) = \int_E \underline{\lim}(g - f_n) \leq \underline{\lim} \int_E (g - f_n) = \int_E g + \underline{\lim}(-\int_E f_n).$$

So $\displaystyle\int_E \underline{\lim}(-f_n) \leq \underline{\lim}(-\int_E f_n)$, that is $\displaystyle\int_E \overline{\lim} f_n \geq \overline{\lim} \int_E f_n$.

Hence $\displaystyle\int_E \underline{\lim}\, f_n \leq \underline{\lim} \int_E f_n \leq \overline{\lim} \int_E f_n \leq \int_E \overline{\lim} f_n$. ◀

6.* (3rd: P.93, Q14; 4th: P.90, Q33 for part b)

(a) Show that under the hypotheses of Theorem 17 (3rd. ed.) (i.e. g_n, g are integrable such that $g_n \to g$ pointwisely a.e., f_n are measurable, $|f_n| \leq g_n$, $f_n \to f$ pointwisely a.e. and $\int g = \lim \int g_n$) we have $\int |f_n - f| \to 0$.

(b) Let $\{f_n\}$ be a sequence of integrable functions such that $f_n \to f$ a.e. with f integrable. Then $\int |f_n - f| \to 0$ if and only if $\int |f_n| \to \int |f|$.

Solution. (a) Since $|f_n - f| \leq |f_n| + |f| \leq g_n + g$, $\{g_n + g - |f_n - f|\}$ is a sequence of nonnegative measurable functions. By the assumptions, $\lim_n (g_n + g - |f_n - f|) = 2g$ a.e. By Fatou's Lemma,

$$\int 2g \leq \liminf_n \int (g_n + g - |f_n - f|) = \int 2g - \limsup_n \int |f_n - f|.$$

So $0 \leq \limsup_n \int |f_n - f| \leq 0$ since g is integrable. Therefore $\int |f_n - f| \to 0$.

(b) If $\int |f_n - f| \to 0$, then

$$\left| \int |f_n| - \int |f| \right| \leq \int \big| |f_n| - |f| \big| \leq \int |f_n - f| \to 0.$$

Conversely, suppose $\int |f_n| \to \int |f|$. Take $g_n := |f_n|$ and $g := |f|$. Then $g_n \to g$ a.e. and $|f_n| \leq g_n$ for all n. It follows from (a) that $\lim_n \int |f_n - f| = 0$.

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