MATH4050 Real Analysis
Assignment 8

There are 6 questions in this assignment. The page number and question number for each question correspond to that in Royden's Real Analysis, 3rd or 4th edition.

1. (3rd: P.89, Q9; 4th: P.84, Q22)
   Let $\{f_n\}$ be a sequence of nonnegative measurable functions on $(-\infty, +\infty)$ such that $f_n \to f$ a.e., and suppose $\displaystyle\int f_n \to \int f < \infty$. Show that for each measurable set E we have $\displaystyle\int_E f_n \to \int_E f$.

2. (3rd: P.93, Q10)

   a. Show that if f is integrable over E, then so is $|f|$ and
   $$\left|\int_E f\right| \le \int_E |f|.$$
   Does the integrability of $|f|$ imply that of f?

   b. The improper Riemann integral of a function may exist without the function being integrable (in the sense of Lebesgue), e.g., if $f(x) = \dfrac{\sin x}{x}$ on $[0, \infty]$. If f is integrable, show that the improper Riemann integral is equal to the Lebesgue integral when the former exists.

3. (3rd: P.93, Q11)
   If φ is a simple function, we have two definitions for $\int \varphi$, that on page 77 and that on page 90 (3rd. ed.). Show that they are the same. (Note: one definition is the one defining at the first stage, the another one is defined by general Lebesgue integral)

4. (3rd: P.93, Q12; 4th: P.89, Q30)
   Let g be an integrable function on a set E and suppose that $\{f_n\}$ is a sequence of measurable functions such that $|f_n(x)| \le g(x)$ a.e. on E. Show that
   $$\int_E \liminf f_n \le \liminf \int_E f_n \le \limsup \int_E f_n \le \int_E \limsup f_n.$$

5. (3rd: P.93, Q13)
   Let h be an integrable function and $\{f_n\}$ a sequence of measurable functions with $f_n \ge -h$ and $\lim f_n = f$. Show that $\int f_n$ and $\int f$ has a meaning and $\int f \le \liminf \int f_n$.

6. (3rd: P.93, Q14; 4th: P.90, Q33 for part b.)

   a. Show that under the hypotheses of Theorem 17 (3rd. ed.) (i.e. g_n, g are integrable such that $g_n \to g$ pointwisely a.e., f_n are measurable, $|f_n| \le g_n$, $f_n \to f$ pointwisely a.e. and $\displaystyle\int g = \lim \int g_n$) we have $\displaystyle\int |f_n - f| \to 0$.

   b. Let $\{f_n\}$ be a sequence of integrable functions such that $f_n \to f$ a.e. with f integrable. Then $\displaystyle\int |f_n - f| \to 0$ if and only if $\displaystyle\int |f_n| \to \int |f|$.