THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH4030 Differential Geometry
Solution of Assignment 5

1.

$$X(u,v) = (f(v)\cos u, f(v)\sin u, g(v))$$
$$X_u = (-f(v)\sin u, f(v)\cos u, 0)$$
$$X_v = (f^{'}(v)\cos u, f^{'}(v)\sin u, g^{'}(v))$$

The first fundamental form is

$$g_{ij} = \begin{bmatrix} f^2 & 0 \\ 0 & (f^{'})^2 + (g^{'})^2 \end{bmatrix}$$

$$g^{ij} = \begin{bmatrix} \frac{1}{f^2} & 0 \\ 0 & \frac{1}{(f^{'})^2+(g^{'})^2} \end{bmatrix}$$

We have the formula $\Gamma^k_{ij} = \frac{1}{2} g^{kl}(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij})$.
To compute Γ^k_{11}, we can treat it as a vector, i.e.

$$\begin{aligned}\Gamma^k_{11} &= \frac{1}{2} g^{kl}(\partial_1 g_{1l} + \partial_1 g_{1l} - \partial_l g_{11}) \\ &= \frac{1}{2} \begin{bmatrix} \frac{1}{f^2} & 0 \\ 0 & \frac{1}{(f^{'})^2+(g^{'})^2} \end{bmatrix} \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 2ff^{'} \end{bmatrix} \right) \\ &= \begin{bmatrix} 0 \\ -\frac{ff^{'}}{(f^{'})^2+(g^{'})^2} \end{bmatrix}\end{aligned}$$

$$\begin{aligned}\Gamma^k_{12} &= \frac{1}{2} g^{kl}(\partial_1 g_{2l} + \partial_2 g_{1l} - \partial_l g_{12}) \\ &= \frac{1}{2} \begin{bmatrix} \frac{1}{f^2} & 0 \\ 0 & \frac{1}{(f^{'})^2+(g^{'})^2} \end{bmatrix} \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 2ff^{'} \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right) \\ &= \begin{bmatrix} \frac{f^{'}}{f} \\ 0 \end{bmatrix}\end{aligned}$$

$$\begin{aligned}\Gamma^k_{22} &= \frac{1}{2} g^{kl}(\partial_2 g_{2l} + \partial_2 g_{2l} - \partial_l g_{22}) \\ &= \frac{1}{2} \begin{bmatrix} \frac{1}{f^2} & 0 \\ 0 & \frac{1}{(f^{'})^2+(g^{'})^2} \end{bmatrix} \left(\begin{bmatrix} 0 \\ 2f^{'}f^{''} + 2g^{'}g^{''} \end{bmatrix} + \begin{bmatrix} 0 \\ 2f^{'}f^{''} + 2g^{'}g^{''} \end{bmatrix} - \begin{bmatrix} 0 \\ 2f^{'}f^{''} + 2g^{'}g^{''} \end{bmatrix} \right) \\ &= \begin{bmatrix} 0 \\ \frac{f^{'}f^{''}+g^{'}g^{''}}{(f^{'})^2+(g^{'})^2} \end{bmatrix}\end{aligned}$$

2.

$$\begin{aligned}
\frac{d}{dt} < X(t), Y(t) > &=< \frac{d}{dt}X(t), Y(t) > + < X(t), \frac{d}{dt}Y(t) > \\
&=< D_{\alpha'(t)}X(t), Y(t) > + < X(t), D_{\alpha'(t)}Y(t) > \\
&=< (D_{\alpha'(t)}X(t))^{\top} + (D_{\alpha'(t)}X(t))^{\perp}, Y(t) > \\
&\quad + < X(t), (D_{\alpha'(t)}Y(t))^{\top} + (D_{\alpha'(t)}Y(t))^{\perp} > \\
&=< (D_{\alpha'(t)}X(t))^{\top}, Y(t) > + < X(t), (D_{\alpha'(t)}Y(t))^{\top} > \\
&=< \nabla_{\alpha'(t)}X(t), Y(t) > + < X(t), \nabla_{\alpha'(t)}Y(t) >
\end{aligned}$$

Given two parallel vectors fields $X(t), Y(t)$, we have

$$\nabla_{\alpha'(t)}X(t) = \nabla_{\alpha'(t)}Y(t) = 0$$

By the previous computation,

$$\frac{d}{dt} < X(t), X(t) >= \frac{d}{dt} < Y(t), Y(t) >= \frac{d}{dt} < X(t), Y(t) >= 0$$

Hence $|X(t)|, |Y(t)|, < X(t), Y(t) >$ are some constants. The angle between $X(t)$ and $Y(t)$ is $\cos^{-1}\left(\frac{< X, Y >}{|X||Y|}\right)$ which is a constant.

3. The Gauss equation $\partial_k \Gamma^\ell_{ij} - \partial_j \Gamma^\ell_{ik} + \Gamma^p_{ij}\Gamma^\ell_{pk} - \Gamma^p_{ik}\Gamma^\ell_{pj} = g^{\ell p}\left(A_{ij}A_{kp} - A_{ik}A_{jp}\right)$ is a constraint on the first fundament form and the second fundamental form of a surface in $\mathbb{R}^3$.
   Put $i = j = 1, k = l = 2$, we have

$$det\left(A_{ij}\right) = \frac{1}{g^{22}}\left(\partial_2\Gamma^2_{11} - \partial_1\Gamma^2_{12} + \Gamma^p_{11}\Gamma^2_{p2} - \Gamma^p_{12}\Gamma^2_{p1}\right)$$

   By the assumption,

$$g_{ij} = \begin{bmatrix} E & 0 \\ 0 & G \end{bmatrix}$$

   We have

$$\begin{aligned}
\Gamma^k_{11} &= \frac{1}{2}g^{kl}(\partial_1 g_{1l} + \partial_1 g_{1l} - \partial_l g_{11}) \\
&= \frac{1}{2}\begin{bmatrix} \frac{1}{E} & 0 \\ 0 & \frac{1}{G} \end{bmatrix}\left(\begin{bmatrix} E_u \\ 0 \end{bmatrix} + \begin{bmatrix} E_u \\ 0 \end{bmatrix} - \begin{bmatrix} E_u \\ E_v \end{bmatrix}\right) \\
&= \begin{bmatrix} \dfrac{E_u}{2E} \\ \\ -\dfrac{E_v}{2G} \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
\Gamma^k_{12} &= \frac{1}{2}g^{kl}(\partial_1 g_{2l} + \partial_2 g_{1l} - \partial_l g_{12}) \\
&= \frac{1}{2}\begin{bmatrix} \frac{1}{E} & 0 \\ 0 & \frac{1}{G} \end{bmatrix}\left(\begin{bmatrix} 0 \\ G_u \end{bmatrix} + \begin{bmatrix} E_v \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) \\
&= \begin{bmatrix} \dfrac{E_v}{2E} \\ \\ \dfrac{G_u}{2G} \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
\Gamma^k_{22} &= \frac{1}{2}g^{kl}(\partial_2 g_{2l} + \partial_2 g_{2l} - \partial_l g_{22}) \\
&= \frac{1}{2}\begin{bmatrix} \frac{1}{E} & 0 \\ 0 & \frac{1}{G} \end{bmatrix}\left(\begin{bmatrix} 0 \\ G_v \end{bmatrix} + \begin{bmatrix} 0 \\ G_v \end{bmatrix} - \begin{bmatrix} G_u \\ G_v \end{bmatrix}\right) \\
&= \begin{bmatrix} -\dfrac{G_u}{2E} \\ \\ \dfrac{G_v}{2G} \end{bmatrix}
\end{aligned}$$

Then

$$\begin{aligned}
det(A) &= \frac{1}{g^{22}}\left(\partial_2\Gamma^2_{11} - \partial_1\Gamma^2_{12} + \Gamma^p_{11}\Gamma^2_{p2} - \Gamma^p_{12}\Gamma^2_{p1}\right) \\
&= G\left[\left(-\frac{E_v}{2G}\right)_v - \left(\frac{G_u}{2G}\right)_u + \left(\frac{E_u}{2E}\right)\left(\frac{G_u}{2G}\right) + \left(-\frac{E_v}{2G}\right)\left(\frac{G_v}{2G}\right)\right. \\
&\quad \left. - \left(\frac{E_v}{2E}\right)\left(-\frac{E_v}{2G}\right) - \left(\frac{G_u}{2G}\right)\left(\frac{G_u}{2G}\right)\right] \\
&= \frac{G}{2}\left\{-\left(\frac{E_v}{G}\right)_v - \left(\frac{G_u}{G}\right)_u - \frac{1}{2}\left[\frac{G_v}{G} - \frac{E_v}{E}\right]\left(\frac{E_v}{G}\right) - \frac{1}{2}\left[\frac{G_u}{G} - \frac{E_u}{E}\right]\left(\frac{G_u}{G}\right)\right\} \\
&= -\frac{\sqrt{EG}}{2}\left[\left(\sqrt{\frac{G}{E}}\right)\left(\frac{E_v}{G}\right)_v + \left(\sqrt{\frac{G}{E}}\right)\left(\frac{G_u}{G}\right)_u\right. \\
&\quad \left. + \left(\sqrt{\frac{G}{E}}\right)_v\left(\frac{E_v}{G}\right) + \left(\sqrt{\frac{G}{E}}\right)_u\left(\frac{G_u}{G}\right)\right] \\
&= -\frac{\sqrt{EG}}{2}\left\{\left[\left(\sqrt{\frac{G}{E}}\right)\left(\frac{E_v}{G}\right)_v + \left(\sqrt{\frac{G}{E}}\right)_v\left(\frac{E_v}{G}\right)\right]\right. \\
&\quad \left. + \left[\left(\sqrt{\frac{G}{E}}\right)\left(\frac{G_u}{G}\right)_u + \left(\sqrt{\frac{G}{E}}\right)_u\left(\frac{G_u}{G}\right)\right]\right\} \\
&= -\frac{\sqrt{EG}}{2}\left[\left(\frac{E_v}{\sqrt{EG}}\right)_v + \left(\frac{G_u}{\sqrt{EG}}\right)_u\right]
\end{aligned}$$

Thus we have

$$\begin{aligned}
K &= \frac{det(A)}{det(g)} \\
&= -\frac{1}{2\sqrt{EG}}\left[\left(\frac{E_v}{\sqrt{EG}}\right)_v + \left(\frac{G_u}{\sqrt{EG}}\right)_u\right]
\end{aligned}$$

Now suppose $E = G = \lambda(u, v)$,

$$\begin{aligned}
K &= -\frac{1}{2\lambda}\left[\left(\frac{\lambda_v}{\lambda}\right)_v + \left(\frac{\lambda_u}{\lambda}\right)_u\right] \\
&= -\frac{1}{2\lambda}\left[(\log\lambda)_{vv} + (\log\lambda)_{uu}\right] \\
&= -\frac{1}{2\lambda}\Delta(\log\lambda)
\end{aligned}$$

4. (a) By Q3, $K = -\frac{1}{2\lambda}\Delta(\log\lambda) = -\frac{1}{2}\Delta(\log 1) = 0$. But $\dfrac{det(A)}{det(g)} = \dfrac{-1}{1} = -1 \neq 0$.
So there is no surface with such g_{ij} and A_{ij}.

(b) Assume $\cos u \neq 0$, the only non-zero term in $\partial_k g_{ij}$ is $\partial_1 g_{22} = -2\sin u\cos u$. Thus

$$\Gamma^2_{12} = \frac{1}{2\cos^2 u}(-2\sin u\cos u) = -\tan u$$
$$\Gamma^1_{22} = \frac{1}{2}(2\sin u\cos u) = \sin u\cos u$$

Hence

$$\partial_1 A_{22} - \partial_2 A_{21} + \Gamma^p_{22}A_{p1} - \Gamma^p_{21}A_{p2} = \cos^3 u\sin u + \tan u \quad (i = j = 2, k = 1)$$

So g_{ij} and A_{ij} do not satisfy the Gauss equation. It means there is no surface with such g_{ij} and A_{ij}.

5. (a) "$\Leftarrow$"Given that p is a local maximum of $f(x) = |x - p_0|^2$ for $x \in S$.
Hence we have for any $\alpha(t) : (-\epsilon, \epsilon) \to S$ with $\alpha(0) = p$ and $|\alpha'(0)| = 1$, $f(\alpha(t))$ attains a maximum at $t = 0$,

$$0 = \frac{d}{dt}\Big|_{t=0} f(\alpha(t)) = 2 < \alpha'(0), p - p_0 >$$

$$0 \geq \frac{d^2}{dt^2}\Big|_{t=0} f(\alpha(t)) = 2 < \alpha''(0), p - p_0 > +2|\alpha'(0)|^2$$

So we have

$$(p - p_0) \perp T_pS, p - p_0 = cN(p) \quad \text{for some non-zero constant } c$$

$$< \alpha''(0), p - p_0 > \leq -1$$

Let $\{k_1, k_2\}$ and $\{v_1, v_2\}$ be the corresponding eigenvalues and eigenvectors of the shape operator, i.e.

$$-dN(v_i) = k_i v_i$$

Let $\alpha_i(t) : (-\epsilon, \epsilon) \to S$ with $\alpha_i(0) = p$ and $\alpha_i'(0) = v_i$.

$$\begin{aligned}
k_i &=< -dN(v_i), v_i > \\
&=< -dN(\alpha_i'(0)), \alpha_i'(0) > \\
&=< -\nabla_{\alpha_i'(0)} N, \alpha_i'(0) > \\
&=< N, \nabla_{\alpha_i'(0)} \alpha_i'(t) > \\
&=< N, \alpha_i''(0) > \\
&= \frac{1}{c} < p - p_0, \alpha_i''(0) >
\end{aligned}$$

The Gauss curvature at p
$K = k_1 k_2 = [\frac{1}{c^2} < p - p_0, \alpha_1''(0) >< p - p_0, \alpha_2''(0) >] \geq \frac{1}{c^2} > 0$

"$\Rightarrow$" Given that $K > 0$, let $\{k_1, k_2\}$ and $\{v_1, v_2\}$ be the corresponding eigenvalues and eigenvectors of the shape operator where $\{v_1, v_2\}$ is an orthonormal basis of T_pS and N to be chosen such that $k_1 \geq k_2 > 0$.
Let $p_0 = p + aN(p)$ where $a > \dfrac{1}{k_2}$ is a positive constant and $g(x) = |x - p_0|$ defined on S.
Let $\beta(t) : (-\epsilon, \epsilon) \to S$ with $\beta(0) = p$ and $|\beta'(0)| = 1$. So $\beta'(0) = (\cos\theta)v_1 + (\sin\theta)v_2$ for some θ.

$$\begin{aligned}
\frac{d}{dt}\Big|_{t=0} g(\beta(t)) &= 2 < \beta'(0), p - p_0 > \\
&= 2 < \beta'(0), -aN(p) > \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\frac{d^2}{dt^2}\Big|_{t=0} g(\beta(t)) &= 2 < \beta''(0), p - p_0 > +2|\beta'(0)|^2 \\
&= 2 < \beta''(0), -aN(p) > +2 \\
&= -2a < \beta'(0), -dN(\beta'(0)) > +2 \\
&= -2a < (\cos\theta)v_1 + (\sin\theta)v_2, -dN((\cos\theta)v_1 + (\sin\theta)v_2) > +2 \\
&= -2a < (\cos\theta)v_1 + (\sin\theta)v_2, (\cos\theta)k_1v_1 + (\sin\theta)k_2v_2 > +2 \\
&= -2a[(\cos^2\theta)k_1 + (\sin^2\theta)k_2] + 2 \\
&\leq -2ak_2 + 2 \\
&< -2\frac{1}{k_2}k_2 + 2 \\
&= 0
\end{aligned}$$

So $g(x)$ attains a local maximum at $p = 0$ since β is arbitrary.

(b) Suppose S is compact without boundary. So there is a point $p \in S$ such that $|p|^2$ is a global maximum.
By the previous result, $K(p) > 0$, this is a contraction to the assumption.

6.
$$X(u,v) = (u\cos v, u\sin v, \log u)$$
$$X_u = (\cos v, \sin v, \frac{1}{u})$$
$$X_v = (-u\sin v, u\cos v, 0)$$
$$g_{ij} = \begin{bmatrix} 1+\frac{1}{u^2} & 0 \\ 0 & u^2 \end{bmatrix}$$

By Q3, the Gauss curvature is

$$\begin{aligned} K &= -\frac{1}{2\sqrt{1+u^2}}\left[\left(\frac{0}{\sqrt{1+u^2}}\right)_v + \left(\frac{2u}{\sqrt{1+u^2}}\right)_u\right] \\ &= -\frac{1}{[1+u^2]^2} \end{aligned}$$

$$\tilde{X}(u,v) = (u\cos v, u\sin v, v)$$
$$\tilde{X}_u = (\cos v, \sin v, 0)$$
$$\tilde{X}_v = (-u\sin v, u\cos v, 1)$$
$$\tilde{g_{ij}} = \begin{bmatrix} 1 & 0 \\ 0 & 1+u^2 \end{bmatrix}$$

By Q3, the Gauss curvature is

$$\begin{aligned} \tilde{K} &= -\frac{1}{2\sqrt{1+u^2}}\left[\left(\frac{0}{\sqrt{1+u^2}}\right)_v + \left(\frac{2u}{\sqrt{1+u^2}}\right)_u\right] \\ &= -\frac{1}{[1+u^2]^2} \\ &= K \end{aligned}$$

So they have the same Gauss curvature. But $\tilde{X} \circ X^{-1}$ is not an isometry since they have the different first fundamental forms.

7. From Q1 of hw4, we know that the Gauss curvature of $\{z = x^2 - y^2\}$ is

$$K = -\frac{4}{[1 + 4x^2 + 4y^2]^2} < 0$$

The Gauss curvature of a round sphere is positive and the Gauss curvature of a cylinder is zero.
By the Gauss Egregium Theorem, the Gaussian curvature of a surface is invariant by local isometries. So they are not locally isometric.