**THE CHINESE UNIVERSITY OF HONG KONG**
**Department of Mathematics**
**MATH 4030** (First term, 2019-20)
**Differential Geometry**
**Preliminary materials** [1]

In this short note, we collect some background materials that you are supposed to know for the course MATH 4030. This would serve as (i) a review of some concepts and theorems you have learned before that are relevant to the course; (ii) a guide towards references where you can look up more detailed treatments. Sections marked with an asterisk $*$ are only required for specific topics in the course and you can just assume them as facts.

Throughout this course, we use $\mathbb{R}^n$ to denote the space of $n$-tuples of real numbers

$$\mathbb{R}^n := \{(x^1, \cdots, x^n) : x_i \in \mathbb{R} \text{ for } i = 1, \cdots, n\}$$

equipped with the norm

$$|(x^1, \cdots, x^n)| := \sqrt{(x^1)^2 + (x^2)^2 + \cdots + (x^n)^2}.$$

The norm is induced by the standard dot product

$$\langle (x^1, \cdots, x^n), (y^1, \cdots, y^n) \rangle := \sum_{i=1}^n x^i y^i$$

which makes $(\mathbb{R}^n, \langle \cdot, \cdot \rangle)$ an inner product space.

## 1. Multivariable calculus

We require a solid background of the materials covered in MATH 2010 and 2020 (or their equivalence). A good reference is Chapters 1 to 3 of Michael Spivak's *Calculus on Manifolds* and Appendix of Chapter 2 in the textbook.

### 1.1. **Functions on Euclidean space.**

Let $U \subset \mathbb{R}^n$. A **function** $F : U \to \mathbb{R}^m$ is a rule which assign each point $x \in U$ some point $F(x) \in \mathbb{R}^m$. The set $U$ is called the **domain** of $F$. For $V \subset U$ and $A \subset \mathbb{R}^m$, we denote the **image** of $V$ and **pre-image** of $A$ under $F$ respectively by

$$F(V) := \{F(x) : x \in V\} \subset \mathbb{R}^m \qquad \text{and} \qquad F^{-1}(A) := \{x \in U : F(x) \in A\} \subset \mathbb{R}^n.$$

The notation $F : U \to A$ means $F(U) \subset A$.One can express a function $F : U \to \mathbb{R}^m$ as

$$F(x) = (f^1(x), \cdots, f^m(x)),$$

where each **component function** $f^i : U \to \mathbb{R}$ is a real-valued function defined on $U$. Given two functions $F : U \to \mathbb{R}^m$ and $G : V \to \mathbb{R}^p$ where $F(U) \subset V \subset \mathbb{R}^m$, one defines the **composition** $G \circ F : U \to \mathbb{R}^p$ by $G \circ F(x) := G(F(x))$. A function $F : U \to A$ is said to be **1-1** if $F(x) \neq F(y)$ whenever $x \neq y$; **onto** if $F(U) = A$; **bijective** if it is both 1-1 and onto.

A function $F : U \to \mathbb{R}^m$ is **continuous** at $a \in U$ if $\lim_{x \to a} F(x) = F(a)$, i.e. for any $\epsilon > 0$, there exists $\delta > 0$ such that $|F(x) - F(a)| < \epsilon$ for every $x \in U$ such that $|x - a| < \delta$. We simply say $F$ is continuous if it is continuous at each $a \in U$. A function $F : U \to \mathbb{R}^m$ is continuous if and only if each component function $f^i$ is continuous. Compositions of continuous functions are continuous. Restrictions of continuous functions to any sub-domain are still continuous. A bijective continuous function $F : A \to B$ is said to be a **homeomorphism** if its inverse $F^{-1} : B \to A$ is also continuous.

### 1.2. **Differentiation.**

Let $U \subset \mathbb{R}^n$ be an *open* set. A function $F : U \to \mathbb{R}^m$ is **differentiable** at $a \in U$ if there exists a (unique) linear map $dF_a : \mathbb{R}^n \to \mathbb{R}^m$, called the **differential** of $F$ at $a$ such that

$$\lim_{h \to 0} \frac{|F(a + h) - F(a) - dF_a(h)|}{|h|} = 0.$$

We simply say $F$ is differentiable if it is differentiable at each $a \in U$. For a function $F : A \to \mathbb{R}^m$ where $A \subset \mathbb{R}^n$ is an arbitrary subset (*not necessarily open!*), we say that it is differentiable if there exists an open set $U \subset \mathbb{R}^n$ containing $A$ and a differentiable function $\overline{F} : U \to \mathbb{R}^m$ such that the restriction $\overline{F}|_A = F$. Compositions of differentiable functions are differentiable. Moreover, their differentials satisfy the **Chain Rule**:

$$d(G \circ F)_a = dG_{F(a)} \circ dF_a.$$

[1] Last updated September 3, 2019

The differential $dF_a : \mathbb{R}^n \to \mathbb{R}^m$ can be expressed using the standard bases of $\mathbb{R}^n$ and $\mathbb{R}^m$ by the **Jacobian matrix**

$$DF(a) = \begin{pmatrix} \frac{\partial f^1}{\partial x^1} & \cdots & \frac{\partial f^1}{\partial x^n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f^m}{\partial x^1} & \cdots & \frac{\partial f^m}{\partial x^n} \end{pmatrix}$$

where $\frac{\partial f^i}{\partial x^j}$ denotes the $j$-th **partial derivative** of the $i$-th component function $f^i$, evaluated at $a$. Recall that a function $f : U \to \mathbb{R}$ is said to be of **class** $C^k$ if all the partial derivatives of $f$ up to order $k$ exist and are continuous. If $f$ is of class $C^k$ for every $k \in \mathbb{N}$, we say $f$ is **smooth** or **of class** $C^\infty$.

**Theorem 1** (Clairaut's Theorem). *Let $f : U \to \mathbb{R}$ be a $C^k$ function. Then all mixed partial derivatives up to order $k$ are independent of the order of taking the derivatives. For instance, for any $C^2$ function $f(x^1, x^2) : U \to \mathbb{R}$, we have*

$$\frac{\partial^2 f}{\partial x^1 \partial x^2} = \frac{\partial^2 f}{\partial x^2 \partial x^1}.$$

We list below some useful theorems about differentiation of a real-valued function.

**Theorem 2.** *Let $f : U \subset \mathbb{R}^n \to \mathbb{R}$ be a differentiable function defined on an open subset $U \subset \mathbb{R}^n$.*

(i) *At each $p \in U$ which is a local minima or maxima of $f$, we have $df_p = 0$;*
(ii) *If $df_p = 0$ at every point $p \in U$ and $U$ is connected, then $f$ is constant on $U$.*

A vector-valued function $F : U \to \mathbb{R}^m$ is said to be of class $C^k$ if each component function is of class $C^k$.

**Theorem 3** (Inverse Function Theorem). *Let $F : U \subset \mathbb{R}^n \to \mathbb{R}^n$ be a $C^1$ function. Suppose $dF_a : \mathbb{R}^n \to \mathbb{R}^n$ is an isomorphism. Then there exists a neighborhood $V$ of $a$ in $U$ and a neighborhood $W$ of $F(p)$ in $\mathbb{R}^n$ such that $F : V \to W$ is a* ***diffeomorphism*** *(i.e. $F$ has a differentiable inverse $F^{-1} : W \to V$).*

**Theorem 4** (Implicit Function Theorem). *Let $F : U \subset \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ be a $C^1$ function. Suppose $F(a, b) = 0$ at some $(a, b) \in U$ and that the $m \times m$ matrix*

$$\begin{pmatrix} \frac{\partial f^1}{\partial x^{n+1}} & \cdots & \frac{\partial f^1}{\partial x^{n+m}} \\ \vdots & \ddots & \vdots \\ \frac{\partial f^m}{\partial x^{n+1}} & \cdots & \frac{\partial f^m}{\partial x^{n+m}} \end{pmatrix}$$

*is invertible. Then there exist a neighborhood $A \subset \mathbb{R}^n$ of $a$ and a neighborhood $B \subset \mathbb{R}^m$ of $b$ such that $A \times B \subset U$ and $F(x, g(x)) = 0$ for some differentiable function $g : A \to B$.*

1.3. **Integration.** Since all the functions we are going to considered in this course are at least *continuous*, we do not go into detail about various definitions of integrability (e.g. Riemann or Lebesgue) as all of them are equivalent in this case. For simplicity, we assume that all functions are continuous real-valued functions of two variables.

Let $f : R = [a, b] \times [c, d] \to \mathbb{R}$ be a continuous function. Then the **double integral** of $f$ over the rectangle $R$ can be computed by either of the **iterated integrals** using *Fubini's theorem*:

$$\iint_R f \, dA = \int_c^d \int_a^b f(x, y) \, dxdy = \int_a^b \int_c^d f(x, y) \, dydx.$$

Recall that a subset $A \subset \mathbb{R}^2$ has **measure zero** if for every $\epsilon > 0$, there is a (possibly countable) cover $\{R_1, R_2, \cdots\}$ of $A$ by closed rectangles such that $\sum_{i=1}^\infty \text{area}(R_i) < \epsilon$. It is known that integrals are not affected by sets of measure zero (provided that $f$ is continuous), e.g.

$$\iint_R f \, dA = \iint_{R \setminus A} f \, dA.$$

**Theorem 5** (Change of variable formula). *Let $A \subset \mathbb{R}^2$ be an open set and $\varphi : A \to \mathbb{R}^2$ be a $C^1$ function which is a diffeomorphism onto its image $\varphi(A)$. Then*

$$\iint_{\varphi(A)} f \, dA = \iint_A (f \circ \varphi) |\det d\varphi| \, dA.$$

1.4. **Fundamental Theorems of Calculus.** We recall here various versions of the fundamental theorems of calculus, which say that differentiation and integration are roughly inverse to each other.

**Theorem 6** (1D Fundamental Theorem of Calculus)**.** *Let $f : [a, b] \to \mathbb{R}$ be a $C^1$ function. Then*

$$\int_a^b f'(x)\, dx = f(b) - f(a).$$

In dimension two, we have the following.

**Theorem 7** (Green's Theorem)**.** *Let $P, Q : \Omega \to \mathbb{R}$ be $C^1$ functions on a bounded domain $\Omega \subset \mathbb{R}^2$ with piecewise $C^1$ boundary $\partial\Omega$, with positive orientation. Then*

$$\iint_\Omega \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy = \oint_{\partial\Omega} (P\, dx + Q\, dy).$$

## 2. Linear Algebra

We require a solid background of the materials covered in MATH 1030 and 2040 (or their equivalence). A good reference is Chapters 1 to 6 of Friedberg, Insel and Spence's *Linear Algebra*.

2.1. **Linear maps and matrices.** Let $V, W$ be finite dimensional vector spaces over $\mathbb{R}$. A function $T : V \to W$ is called a **linear map** if $T(au + bv) = aT(u) + bT(v)$ for any $u, v \in V$, $a, b \in \mathbb{R}$. For any fixed bases (i.e. linear independent spanning set) $\beta = \{v_1, \cdots, v_n\}$ of $V$ and $\gamma = \{w_1, \cdots, w_m\}$ of $W$, we can represent $T$ by an $m \times n$ matrix

$$[T]_\beta^\gamma = \begin{pmatrix} a_1^1 & \cdots & a_1^n \\ \vdots & \ddots & \vdots \\ a_m^1 & \cdots & a_m^n \end{pmatrix}$$

where $T(u_i) = \sum_{j=1}^m a_i^j w_j$. When $V = W$ and $\beta = \gamma$, one simply writes $[T]_\beta$ or sometimes even $T$ when the basis $\beta$ is understood. If $T : V \to V$ is a linear map and $\beta, \gamma$ are two different bases of the same vector space $V$, then the matrices $[T]_\beta$ and $[T]_\gamma$ are related by the **change of basis formula**:

$$[T]_\gamma = Q^{-1}[T]_\beta Q$$

where $Q = [I]_\gamma^\beta$ is invertible. We define two important invariants, called the **determinant** and **trace** of $T$ respectively, by $\det T = \det[T]_\beta$ and $\operatorname{tr} T = \operatorname{tr}[T]_\beta$. By the change of basis formula, these are independent of the choice of the basis $\beta$.

Let $T : V \to V$ be a linear map. A non-zero vector $v \in V$ is said to be an **eigenvector** of $T$ if $Tv = \lambda v$ for some $\lambda \in \mathbb{R}$. The scalar $\lambda \in \mathbb{R}$ is called the **eigenvalue** of $T$ associated to $v$. Every eigenvalue $\lambda \in \mathbb{R}$ of $T$ is a (real) solution to the **characteristic equation**

$$\det([T]_\beta - \lambda I) = 0$$

for *any* choice of basis $\beta$ for $V$. A linear map $T : V \to V$ is **diagonalizable** if there exists a basis consisting of eigenvectors of $T$.

2.2. **Self-adjoint linear maps and bilinear forms.** Let $V$ be an $n$-dimensional vector space. A map $B : V \times V \to \mathbb{R}$ is called a **bilinear form** on $V$ if it is a linear function in each variable, i.e. for all $\lambda_1, \lambda_2 \in \mathbb{R}$ and $v_1, v_2, w \in V$, we have

(i) $B(\lambda_1 v_1 + \lambda_2 v_2, w) = \lambda_1 B(v_1, w) + \lambda_2 B(v_2, w)$;
(ii) $B(w, \lambda_1 v_1 + \lambda_2 v_2) = \lambda_1 B(w, v_1) + \lambda_2 B(w, v_2)$.

A bilinear form $B$ is **symmetric** if $B(v, w) = B(w, v)$ for all $v, w \in V$. Given any basis $\beta = \{v_1, \cdots, v_n\}$ of $V$, we can express $B$ as an $n \times n$ matrix:

$$[B]_\beta = \begin{pmatrix} B(v_1, v_1) & \cdots & B(v_1, v_n) \\ \vdots & \ddots & \vdots \\ B(v_n, v_1) & \cdots & B(v_n, v_n) \end{pmatrix}.$$

Note that $[B]_\beta$ is a symmetric matrix if $B$ is symmetric. Moreover, if $\beta, \gamma$ are two different bases of $V$, then we have the following transformation law (*Notice how this is different from that for linear operators!*):

$$[B]_\gamma = Q^t [B]_\beta Q.$$

An **inner product** on $V$ is a symmetric bilinear form $\langle \cdot, \cdot \rangle$ on $V$ such that $\langle v, v \rangle > 0$ for all non-zero $v \in V$. The **length** of $v$ is then defined to be $|v| := \langle v, v \rangle^{1/2}$. Two vectors $v, w \in V$ are said to be **perpendicular** or **orthogonal** if $\langle v, w \rangle = 0$. A basis $\{v_1, \cdots, v_n\}$ is said to be an **orthonormal basis** if

$$\langle v_i, v_j \rangle = \delta_{ij} := \begin{cases} 1 & \text{when } i = j, \\ 0 & \text{when } i \neq j. \end{cases}$$

Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space. A linear map $T : V \to V$ is **self-adjoint** if $\langle Tv, w \rangle = \langle v, Tw \rangle$ for all $v, w \in V$. If $\beta$ is an orthonormal basis of $(V, \langle \cdot, \cdot \rangle)$, then the matrix representation of $T$ is *symmetric*, i.e. $[T]_\beta = [T]_\beta^t$.

**Theorem 8** (Spectral Theorem)**.** *For any self-adjoint operator $T : V \to V$ on a finite dimensional inner product space $(V, \langle \cdot, \cdot \rangle)$, there exists an orthonormal basis $\beta$ of $V$ consisting of eigenvectors of $T$. Equivalently, for any symmetric real $n \times n$ matrix $A$, there exists an orthogonal matrix $P \in O(n)$ (i.e. $P^{-1} = P^t$) such that $PAP^t$ is a diagonal matrix.*

On a inner product space $(V, \langle \cdot, \cdot \rangle)$, symmetric bilinear forms $B$ and self-adjoint operators $T$ are naturally identified with each other by the relationship

$$B(v, w) = \langle Tv, w \rangle.$$

2.3. **Isometries of Euclidean spaces.** Recall that an **isometry** on $\mathbb{R}^n$ is a function $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ so that distances are preserved, i.e. $|\varphi(x) - \varphi(y)| = |x - y|$ for any $x, y \in \mathbb{R}^n$. We recall the **orthogonal group** and **special orthogonal group**

$$O(n) := \{ \, n \times n \text{ matrix } A \; : \; AA^t = I\},$$

$$SO(n) := \{A \in O(n) \; : \; \det A > 0\}.$$

Note that $\det A = \pm 1$ for any $A \in O(n)$.

**Theorem 9** (Euclidean isometries)**.** *Any isometry $\varphi$ of $\mathbb{R}^n$ can be written in the form*

$$\varphi(x) = Ax + c$$

*for some $A \in O(n)$ and $c \in \mathbb{R}^n$. Hence, any isometry on $\mathbb{R}^n$ is a composition of rotations, reflections and translations. In case $\varphi$ is orientation-preserving, one can take $A \in SO(n)$ and write any isometry as a composition of rotations and translations.*

## 3. Topology*

A working knowledge of the materials covered in MATH 3070 (or its equivalence) is helpful for the course. A good reference is Chapters 1 of James Munkres' *Analysis on Manifolds* and Chapter 12 of *Topology* by the same author.

3.1. **Compact and connected subspaces of $\mathbb{R}^n$.** Recall that a subset $K \subset \mathbb{R}^n$ is said to be **compact** if any open cover of $K$ has a finite sub-cover. It is known that $K \subset \mathbb{R}^n$ is compact if and only if it is closed and bounded.

**Theorem 10** (Extreme value theorem)**.** *Any continuous function $F : K \to \mathbb{R}$ defined on a compact subset $K \subset \mathbb{R}^n$ has a global maximum and minimum.*

Recall that a metric (or topological) space $X$ is **connected** if $\emptyset$ and $X$ are the only subsets which are both open and closed.

**Theorem 11** (Intermediate value theorem)**.** *Let $F : U \subset \mathbb{R}^n \to \mathbb{R}^m$ be a continuous function. If $X \subset U$ is connected, then $F(X)$ is also connected.*

3.2. **Classification of compact orientable surfaces.** (You can skip this part for now and get back to it later in the course.) Let $M \subset \mathbb{R}^3$ be c compact, oriented surface (possibly with piecewise-smooth boundary). It is know that any such surface admits a **triangulation** $\tau$, i.e. we may write

$$M = \bigcup_{i=1}^{m} \Delta_i$$

such that

(i) each $\Delta_i$ is the image of a triangle under an orientation-preserving orthogonal parametrization;
(ii) $\Delta_i \cap \Delta_j$, $i \neq j$, is either empty, a single vertex, or a single edge;
(iii) when $\Delta_i \cap \Delta_j$ consists of a single edge, the orientation of the edge are opposite in $\Delta_i$ and $\Delta_j$;
(iv) at most one edge of $\Delta_i$ is contained in in the boundary of $M$.

Given any triangulation $\tau$ of $M$ as above, we define the **Euler characteristic** of $M$ (with respect to $\tau$) as

$$\chi(M, \tau) := V - E + F$$

where $V, E$ and $F$ are the number of vertices, edges and faces respectively of the triangulation $\tau$. It is an amazing fact that $\chi$ is a *topological invariant*, i.e. the number $\chi(M, \tau)$ is independent of the triangulation $\tau$. Moreover, when $\partial M = \emptyset$, the Euler characteristic gives a complete set of invariants.

**Theorem 12** (Classification of closed orientable surfaces)**.** *For each non-negative integer $g$, there is exact one (up to homeomorphism) closed orientable surface $\Sigma_g$ with $\chi(\Sigma_g) = 2 - 2g$.*

## 4. Differential Equations*

A working knowledge of the materials covered in MATH 3270 (or its equivalence) is helpful for the course. A good reference is Chapters 2 and 7 of Boyce and DiPrima's *Elementary differential equations and boundary value problems.*

**Theorem 13** (Fundamental Theorem of ODEs)**.** *Let $U \subset \mathbb{R}^n$ be an open subset and $I \subset \mathbb{R}$ be an open interval containing $0$. Suppose $x_0 \in U$. If $F : U \times I \to \mathbb{R}^n$ is Lipschitz in $x$ (i.e. there exists a constant $C > 0$ such that $|F(x,t) - F(y,t)| \leq C|x - y|$ for all $x, y \in U$, $t \in I$), then the differential equation*

$$\frac{dx}{dt} = F(x,t), \qquad x(0) = x_0$$

*has a unique solution $x = x(t, x_0)$ defined for all $t$ in some sub-interval $I' \subset I$ containing $0$. Moreover, if $F$ is $C^k$, then $x$ is $C^k$ as a function of both $t$ and the initial condition $x_0$.*

In the case of linear equations, we have a stronger global result.

**Theorem 14** (Global existence for first order linear ODEs)**.** *Suppose $A(t)$ is a continuous $n \times n$ matrix function defined on an interval $I \subset \mathbb{R}$. Then the differential equation*

$$\frac{dx}{dt} = A(t)x(t), \qquad x(0) = x_0$$

*has a unique solution on the whole interval $I$.*