MATH 4030 Problem Set 3[1]
Due date: Oct 22, 2019

Reading assignment: do Carmo's Section 2.3, 2.4, 2.5, 2.6

Problems: (Those marked with † are optional.)

1. Let $f, g : \mathbb{R} \to \mathbb{R}$ be smooth functions such that $f(u) > 0$ and $g'(u) > 0$ for all $u \in \mathbb{R}$. Define a smooth map $X : \mathbb{R} \times (-\pi, \pi) \to \mathbb{R}^3$ by
$$X(u, v) := (f(u) \cos v, f(u) \sin v, g(u)).$$
Show that the image $S := X(\mathbb{R} \times (-\pi, \pi))$ is a surface, and that all the normal lines of S pass through the z-axis. What does the surface S look like?

2. If all normal lines to a connected surface S passes through a fixed point $p_0 \in \mathbb{R}^3$, show that S is contained in a sphere.

3. Let S be a compact surface, show that there exists a straight line ℓ cutting S orthogonally at at least two points. *(Hint: Consider two points on S which are of maximal distance apart.)*

4. Let $S = \{p \in \mathbb{R}^3 : |p|^2 - \langle p, a\rangle^2 = r^2\}$ with $|a| = 1$ and $r > 0$ be a right cylinder of radius r whose axis is the line passing through the origin with direction a. Prove that
$$T_pS = \{v \in \mathbb{R}^3 : \langle p, v\rangle - \langle p, a\rangle\langle a, v\rangle = 0\}.$$
Conclude that all the normal lines of S cut the axis orthogonally. Prove the converse as well: i.e. if S is a connected surface whose normal lines all intersect a fixed straight line $\ell \subset \mathbb{R}^3$ orthogonally, then S is a subset of a right cylinder with axis ℓ.

5. Let $S \subset \mathbb{R}^3$ be a surface.
   (a) Fix a point $p_0 \in \mathbb{R}^3$ such that $p_0 \notin S$, show that the *distance function from* p_0
   $$f(p) := |p - p_0|$$
   defines a smooth function $f : S \to \mathbb{R}$. Moreover, prove that $p \in S$ is a critical point of f if and only if the line joining p to p_0 is normal to S at p.
   (b) Fix a unit vector $v \in \mathbb{R}^3$, show that the *height function along* v
   $$h(p) := \langle p, v\rangle$$
   defines a smooth function $h : S \to \mathbb{R}$. Prove that $p \in S$ is a critical point of h if and only if v is normal to S at p.

6. Find the area of the torus of revolution S defined by
$$S = \{(x, y, z) \in \mathbb{R}^3 : (\sqrt{x^2 + y^2} - a)^2 + z^2 = r^2\},$$
where $a > r > 0$ are given positive constants.

7. Let $a \in \mathbb{R}$ be a regular value of a smooth function $F : O \subset \mathbb{R}^3 \to \mathbb{R}$. Prove that the surface $S = F^{-1}(a)$ is orientable.

[1]Last revised on October 8, 2019

8. Compute the first fundamental form and surface area of the helicoid parametrized by

$$X(u,v) := (u\cos v, u\sin v, bv) \qquad 1 < u < 3,\ 0 < v < 2\pi.$$

9. (†) (*Change of coordinates for tangent vectors*) Let $p \in S$ be a point on the surface S. Suppose there are two parametrizations

$$X(u,v) : U \subset \mathbb{R}^2 \to V \subset S,$$
$$\overline{X}(\overline{u},\overline{v}) : \overline{U} \subset \mathbb{R}^2 \to \overline{V} \subset S$$

such that $p \in V \cap \overline{V}$. Let $\psi = \overline{X}^{-1} \circ X : X^{-1}(V \cap \overline{V}) \to \overline{X}(V \cap \overline{V})$ be the transition map which can be written in (u,v) and $(\overline{u},\overline{v})$ coordinates as

$$\psi(u,v) = (\overline{u}(u,v), \overline{v}(u,v)).$$

If $v \in TpS$ is a tangent vector which can be expressed in local coordinates as

$$a_1\frac{\partial X}{\partial u} + a_2\frac{\partial X}{\partial v} = v = b_1\frac{\partial \overline{X}}{\partial \overline{u}} + b_2\frac{\partial \overline{X}}{\partial \overline{v}},$$

where the partial derivatives are evaluated at the point $X^{-1}(p)$ and $\overline{X}^{-1}(p)$ respectively, show that

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} \frac{\partial \overline{u}}{\partial u} & \frac{\partial \overline{u}}{\partial v} \\ \frac{\partial \overline{v}}{\partial u} & \frac{\partial \overline{v}}{\partial v} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix},$$

where the partial derivatives are evaluated at the point $(u_0, v_0) = X^{-1}(p)$. In other words, the tangent vector transforms by multiplying the *Jacobian matrix* of the transition map ψ.

10. (†) Show that each of the subset $(a, b, c, \neq 0)$ below are surfaces

$$S_1 = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = ax\},$$
$$S_2 = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = by\},$$
$$S_3 = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = cz\}.$$

Prove that they all intersect orthogonally.

11. (†) Construct an explicit diffeomorphism between the ellipsoid $S_1 = \{(x,y,z) \in \mathbb{R}^3 : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1\}$ and the sphere $S_2 = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$.