MATH 4030 Problem Set 2[1]
Due date: Oct 4, 2019

Reading assignment: do Carmo's Section 1.7, 2.2

Problems: (Those marked with † are optional.)

1. Let $\alpha : I \to \mathbb{R}^2$ be a regular plane curve described in polar coordinates by $r = r(\theta)$, i.e.

$$\alpha(\theta) = (r(\theta)\cos\theta, r(\theta)\sin\theta), \quad \theta \in I$$

   (a) Show that for any $[a, b] \subset I$, $L_a^b(\alpha) = \int_a^b \sqrt{r(\theta)^2 + r'(\theta)^2}\, d\theta$.

   (b) Show that the curvature at $\theta \in I$ is given by

$$k(\theta) = \frac{2r'(\theta)^2 - r(\theta)r''(\theta) + r(\theta)^2}{[r'(\theta)^2 + r(\theta)^2]^{3/2}}.$$

2. (*Lancret's theorem*) A space curve $\alpha : I \to \mathbb{R}^3$ p.b.a.l. with $k(s) > 0$ for all $s \in I$ is said to be a *generalized helix* if all of its tangent lines make a constant angle with a given direction, i.e. there exists some unit vector $v \in \mathbb{R}^3$ such that $\alpha'(s) \cdot v \equiv$ constant. Prove that α is a helix if and only if there exists a constant $c \in \mathbb{R}$ such that $\tau(s) = ck(s)$ for all $s \in I$. (*Hint: Differentiate the condition $\alpha'(s) \cdot v \equiv$ constant twice.*)

3. Let $\alpha(s) : I \to \mathbb{R}^3$ be a space curve parametrized by arc length with $\kappa(s) > 0$ for all $s \in I$. The normal line to α at $\alpha(s)$ is the line through $\alpha(s)$ with direction vector $\mathbf{N}(s)$. Suppose all the normal lines to α pass through a common fixed point x_0. Show that α is a circle lying on a plane passing through x_0.

4. A regular closed plane curve is said to be *convex* if it lies on one side of its tangent line at each point. It is known (you do not need to prove this!) that a simple closed regular plane curve is convex if and only if we can choose an orientation of the curve such that $k \geq 0$ everywhere.

   (a) Prove that the ellipse $\alpha(t) = (a\cos t, b\sin t)$, $t \in [0, 2\pi]$, where $a > b > 0$ are positive constants, is convex.

   (b) We say that a regular plane curve $\alpha(t)$ has a *vertex* at $\alpha(t_0)$ if $k'(t_0) = 0$. Show that the ellipse defined in (a) has exactly 4 vertices. Draw a picture to show where these vertices are located.

5. Show that the two-sheeted cone $S = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = z^2\}$ is not a surface.

6. Show that $S = \{(x, y, z) \in \mathbb{R}^3 : z = x^2 - y^2\}$ is a surface and that the following are parametrizations for S:

$$X_1(u, v) = (u + v, u - v, 4uv), \quad (u, v) \in \mathbb{R}^2$$

$$X_2(u, v) = (u\cosh v, u\sinh v, u^2), \quad (u, v) \in \mathbb{R}^2, u \neq 0.$$

7. (†) Let $F(x, y, z) = z^2$. Prove that 0 is not a regular value of F but $F^{-1}(0)$ is a surface.

[1] Last revised on September 20, 2019