THE CHINESE UNIVERSITY OF HONG KONG
MATH4010 Tutorial Note 6
Oct 24, 2019

If you find any mistakes or typos, please report them to ypyang@math.cuhk.edu.hk

Fundamental theorems

Open Mapping Theorem. If a continuous linear operator $T : X \to Y$ between Banach spaces X, Y is surjective, then T is an open map.

Proposition (Bounded Inverse Theorem). If T is a bijective continuous linear operator between the Banach spaces X and Y, then the inverse operator is continuous as well.

Example 6.6. Let $X = (l^1, \|\cdot\|_1), Y = (l^1, \|\cdot\|_\infty)$ and let T be the identity map from X onto Y. It can be seen that T is bounded since

$$\|x\|_\infty = \sup_k |x_k| \le \sum_k |x_k| = \|x\|_1, \forall x \in l^1.$$

On the other hand, for $x^{(n)} = e_1 + \cdots + e_n$,

$$\|x^{(n)}\|_1 = n, \quad \|x^{(n)}\|_\infty = 1,$$

so the inverse of T is not bounded. The space Y is not a Banach space. Indeed, consider the sequence

$$y^{(n)} = \left(1, \frac{1}{2}, \cdots, \frac{1}{n}, 0, 0, \cdots\right)$$

in Y. Suppose $y^{(n)} \to y = (y_1, y_2, \cdots)$ in Y. Then

$$\|y^{(n)} - y\|_\infty = \sup_n \left\{ \max_{1\le k\le n} \left| y_k - \frac{1}{k} \right|, |x_{k+1}|, \cdots \right\} \to 0$$

as $n \to \infty$. It follows that $x = (1, 1/2, \cdots, 1/n, \cdots)$, which is not in l^1.

Example 6.7. Let $X = C[0, 1]$ and $Y = \{x \in C^1[0, 1] : x(0) = 0\}$, both equipped with the sup-norm. We define $T : X \to Y$ by

$$(Tx)(t) = \int_0^t x(u)\,du.$$

Then T is bounded since

$$\|Tx\| \le \sup_{u\in[0,1]} |x(u)| = \|x\|.$$

The inverse operator $T^{-1} : Y \to X$ is the differentiation operator $\frac{d}{dt}$, which is unbounded.

To see that Y is not a Banach space, we can consider $f_n = \sqrt{x + \frac{1}{n^2}} - \frac{1}{n}$.

Closed Graph Theorem. If X and Y are Banach spaces and $T : X \to Y$ is a linear operator, then T is continuous if and only if its graph $G(T) = \{(x, Tx) : x \in X\}$ is closed in $X \times Y$, that is, if $\{x_n\}$ is a sequence in X that converges to $x \in X$ and $\{Tx_n\}$ converges to y in Y, then $Tx = y$.

Example 6.9. Let $X = C^1[0,1]$ and $Y = C[0,1]$ both with the sup-norm. The differentiation operator $T = d/dt$ maps X onto Y. This operator is unbounded.

We also claim that $G(T)$ is closed. Let $\{(f_n, f_n')\}$ be a sequence in $G(T)$ which converges to (f, g) in the space $X \times Y$. There are several different ways to introduce a norm in $X \times Y$ such that

$$\|(x_n, y_n)\| \to 0 \Leftrightarrow \|x_n\| \to 0 \text{ and } \|y_n\| \to 0, \quad \forall \{(x_n, y_n)\} \subset X \times Y.$$

A norm satisfying such condition is called a product norm. Usual product norms include

$$\|(x,y)\| = \|x\| + \|y\|, \quad \|(x,y)\| = \max\{\|x\|, \|y\|\}, \quad \|(x,y)\| = \sqrt[p]{\|x\|^p + \|y\|^p}, p > 1.$$

Then f_n converges to f and $Tf_n = f_n'$ converges to g in the sup-norm. So $f_n \to f$ and $f_n' \to g$ uniformly. Hence we have that $Tf = f' = g$. Hence $G(T)$ is closed. Therefore, $(f_n, f_n') \to (f, f')$ in $G(T)$. It follows that $G(T)$ is closed.

Uniform Boundedness Theorem. Let X be a Banach space and Y a normed space. Suppose that $\{T_i : i \in I\}$ is a collection of continuous linear operator from X to Y. If

$$\sup_{i \in I} \|T_i x\| < \infty, \quad \forall x \in X,$$

then

$$\sup_{i \in I} \|T_i\| < \infty.$$

Proposition. Let $A \subset X$. If $f(A)$ is bounded for any $f \in X^*$, then A is bounded.

Example. Let

$$X = \{p(x) = a_0 + a_1 x + \cdots + a_d x^d | a_i \in \mathbb{K}, d \in \mathbb{N}\}$$

be the space of polynomials equipped with the norm $\|p(x)\| = \max_i |a_i|$. We give an example of a sequence of linear maps $T_n : X \to \mathbb{F}$ which are pointwise bounded but not uniformly bounded. Let

$$T_n(p) = a_0 + \cdots + a_{n-1}.$$

We can see that

$$|T_n(p)| \leq |a_0| + \cdots + |a_{n-1}| \leq n\|p\|,$$

so that $\|T\| \leq n$. In fact, this estimates can be improved as

$$|T_n(p)| \leq d\|p\|.$$

This show that the sequence $\{T_n(p)\}$ is bounded for every $p \in X$.

However, we claim that $\|T_n\| = n \to \infty$ by taking $p(x) = 1 + x + x^2 + \cdots + x^{n-1}$.

The Uniform Bounded Theorem fails here because X is not a Banach space.