THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH 3030 Abstract Algebra 2019-20
Homework 6 Solution
Due Date: 24th October 2019

Compulsory part

1. $\Rightarrow$: Suppose that the action is not faithful. Then there is an element $e \neq g \in G$ such that $gx = x$ for all $x \in X$. Since X is a G-set, $ex = x$ for all $x \in X$, so g and e these two distinct elements of G have the same action on each element of X.

   $\Leftarrow$: Suppose there are two distinct elements of G have the same action on each element of X, say g_1 and g_2. One has

   $$\forall x \in X,\ g_1 x = g_2 x \Rightarrow \forall x \in X,\ g_2^{-1} g_1 x = x.$$

   This suggests that the action is not faithful. We are done by contrapositive.

2. We show by the definition.

   - We first have $g_1, g_2 \in G_Y \Rightarrow \forall y \in Y, g_1 y = y$ and $g_2 y = y$. So

     $$g_1 g_2 y = g_1 y = y, \forall y \in Y.$$

     Therefore $g_1 g_2 \in G_Y$.
   - Clearly, the identity of G is also in G_Y.
   - Note that $g \in G_Y \Rightarrow \forall y \in Y, gy = y$. So $\forall y \in Y, g^{-1} y = y$. Hence $g^{-1} \in G_Y$.

3. (a) Because rotation through 0 radians leaves each point of the plane fixed, $0Q = Q$ for all $Q \in \mathbb{R}^2$. For any $Q \in \mathbb{R}^2$, $(\theta_1 + \theta_2)Q = \theta_1(\theta_2)Q$ is also valid, because a rotation counterclockwise through $\theta_1 + \theta_2$ radians can be achieved by sequentially rotating through θ_2 radians and then through θ_1 radians.

   (b) The orbit containing P is a circle centered at the origin $(0, 0)$ with radius the distance from P to the origin.

   (c) It is a cyclic subgroup $\langle 2\pi \rangle$ of G.

4. Note that every G-set is the union of its orbits X_i for $i \in I$. These X_i's are transitive. We are now going to show that X_i is isomorphic to the G-set L of all left cosets of G_{a_i} where $a_i \in X_i$. First of all, for any $x \in X_i$ we have $x = g a_i$ for some $g \in G$, since X_i is transitive and now we are able to define a map $\phi : X_i \to L$ by $\phi(x) = gG_{a_i}$. We need to show that the definition of ϕ is independent of the choice of g: suppose that $g_1 a_i = x_1$ and $g_2 a_i = x_2$, then $g_1 a_i = g_2 a_i \Rightarrow g_2^{-1} g_1 \in G_{a_i}$, so $g_1 G_{a_i} = g_2 G_{a_i}$. It remains to show that it is one-to-one and onto. Suppose $\phi(x_1) = \phi(x_2)$ where $x_1, x_2 \in X_i$ and let $g_1 x_1 = a_i$ and $g_2 x_2 = a_i$ for some $g_1, g_2 \in G$. $\phi(x_1) = \phi(x_2)$ implies that $g_1 G_{a_i} = g_2 G_{a_i}$, so we have $g_1 = g_2 g_0$ for some $g_0 \in G_{a_i}$. $g_1 a_i = x_1$ leads to $g_2 g_0 a_i = x_1$ saying that $g_2 a_i = x_1$ and hence $x_2 = x_1$. This shows that ϕ is one-to-one. For any coset gG_{a_i}, $\phi(g a_i) = gG_{a_i}$

provided that $ga_i \in X_i$ by the transitivity. This shows that ϕ is onto. Finally, to show that ϕ is an isomorphism of G-sets, that is $g\phi(x) = \phi(gx)$. Let $x = g_1 a_i$. Then

$$gx = g(g_1 a_i) = (gg_1)a_i$$

so $\phi(gx) = (gg_1)G_{a_i} = g(g_1 G_{a_i}) = g\phi(x)$.

It is possible that the group G_{a_i} may be the same as the group G_{a_j} for some $i \neq j$ in I, but by attaching the index i to each coset of G_{a_i} and j to each coset ofG_{a_j} as indicated in the statement of this question, we can consider these ith and jth coset G-sets to be disjoint. Identifying X_i with this isomorphic ith coset G-set, we see that X is isomorphic to a disjoint union of left coset G-sets.

5. (a) Let $g \in K$. We have $g(g_0 x_0) = g_0 x_0$, then $(g_0^{-1} g g_0)x_0 = x_0$, which means that $g_0^{-1} g g_0 \in H$, so $g \in g_0 H g_0^{-1}$. Hence $K \subset g_0 H g_0^{-1}$. Making a symmetric argument, starting with $g \in H$, $g_0 x_0$ as initial base point, and obtaining x_0 as second base point by g_0^{-1} acting on $g_0 x_0$, we see that $H \subset g_0^{-1} K g_0$, or equivalently, $g_0 H g_0^{-1} \subset K$. Thus $K = g_0 H g_0^{-1}$.

   (b) Conjecture: The G-set of left cosets of H is isomorphic to the G-set of left cosets of K if and only if H and K are two subgroups of G with conjugation to each other.

   (c) We first show that if H and K are conjugate subgroups of G, then the G-set L_H of left cosets of H is isomorphic to the G-set L_K of left cosets of K. Let $g_0 \in G$ be chosen such that $K = g_0 H g_0^{-1}$. Note that for $aH \in L_H$, we have

$$aHg_0^{-1} = ag g_0^{-1} g_0 H g_0^{-1} = ag_0^{-1} K \in L_K.$$

   We now define $\phi : L_H \to L_K$ by $\phi(aH) = ag_0^{-1}K$. We just saw that $ag_0^{-1}K = (aH)g_0^{-1}$ so the map ϕ is independent of the choice of $a \in H$, that is, ϕ is well defined. Because ag_0^{-1} assumes all values in G as a varies through G, we see that ϕ is onto. If $\phi(aH) = \phi(bH)$, then we have

$$(aH)g_0^{-1} = ag_0^{-1}K = \phi(aH) = \phi(bH) = bg_0^{-1}K = (bH)g_0^{-1},$$

   so $aH = bH$ and ϕ is one to one. To show ϕ is an isomorphism of G-sets, it only remains to show that $\phi(g(aH)) = g\phi(aH)$ for all $g \in G$ and $aH \in L_H$. But $\phi(g(aH)) = \phi((ga)H) = (ga)g_0^{-1}K = g(ag_0^{-1}K) = g\phi(aH)$, and we are done.

   Conversely, suppose that $\phi : L_H \to L_K$ is an isomorphism of the G-set of left cosets of H onto the G-set of left cosets of K. Because ϕ is an onto map, there exists $g_0 \in G$ such that $\phi(g_0 H) = K$. Because ϕ commutes with the action of G, we have

$$(g_0 h g_0^{-1})K = (g_0 h g_0^{-1})\phi(g_0 H) = \phi(g_0 h g_0^{-1} g_0 H) = \phi(g_0 H) = K,$$

   so $g_0 h g_0^{-1} \in K$ for all $h \in H$, that is, $g_0 H g_0^{-1} \subset K$. From $\phi(g_0 H) = K$, we can see that $\phi^{-1}(g_0^{-1}K) = H$, and an argument similar to the one just made then shows that $g_0^{-1} K g_0 \subset H$. Thus $g_0 H g_0^{-1} = K$, that is, the subgroups are indeed conjugate to each other.

6. There are four of them. From the proof of $|Gx| = (G : G_x)$, a transitive G-set is isomorphic to G/G_x (as the set of left cosets). The group S_3 has subgroups $\{Id\}$, $\langle(1,2)\rangle$, $\langle(1,3)\rangle$, $\langle(3,2)\rangle$, $\langle(1,2,3)\rangle$ and S_3. So there are four transitive S_3-set up to isomorphism:

(a) $X = S_3/\{Id\}$;

(b) $X = S_3/\langle(1,2)\rangle \simeq S_3/\langle(1,3)\rangle \simeq S_3/\langle(3,2)\rangle$;

(c) $X = S_3/\langle(1,2,3)\rangle$;

(d) $X = S_3/S_3$.