THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH 3030 Abstract Algebra 2019-20
Tutorial 7
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1. Let G be a finite group, N be a normal subgroup of G and P be a Sylow p-subgroup of G.

   (a) Show that $P \cap N$ is a Sylow p-subgroup of N.

   (b) Show that PN/N is a Sylow p-subgroup of G/N.

Solution. (a) $P \cap N$ is a subgroup of P, so $P \cap N$ is a p-subgroup of N. Also, on the other hand PN is a subgroup of G as $N \lhd G$. By the isomorphism theorem, we have

$$|PN| = |P| \times |N|/|P \cap N|$$

so

$$|N : P \cap N| = |PN : P|.$$

Now P is a Sylow p-subgroup of G, so it is also a Sylow p-subgroup of $PN < G$. Hence

$$p \nmid |PN : P| = |N : P \cap N|,$$

implying that $P \cap N$ is a Sylow p-subgroup of N.

(b) By the isomorphism theorem again, we have

$$|PN : N| = |P : P \cap N|$$

which suggests that PN/N is a p-subgroup of G/N. Furthermore

$$|G/N : PN/N| = \frac{|G|/|N|}{|P|/|P \cap N|} = \frac{|G : P|}{|N : P \cap N|}.$$

Since $p \nmid |G : P|$ as P being a Sylow p-subgroup of G, $p \nmid |G/N : PN/N|$. Hence PN/N is a Sylow p-subgroup of G/N.

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2. Prove that groups of order 56 are not simple.

Solution.
- Let G be a group of order 56. Note that $56 = 2^3 \cdot 7$.
- Consider $n_7(G)$, it is either 1 or 8.

- If $n_7(G) = 1$, then G has a unique Sylow 7-subgroup which is normal. So G is not simple.
- Then $n_7(G) = 8$.
- Observe that the intersection of every pair of Sylow 7-subgroups is trivial (since 7 is prime).
- Hence $(7-1) \times 8 = 48$ elements of G have order 7.
- It means that if P is any Sylow 2-subgroup of G, then P must be contained in the complement of these 48 elements.
- But $56 - 48 = 8$, so there is at most one such P.
- On the other hand, since $n_2(G) \geq 1$, it follows that $n_2(G) = 1$, i.e. G has a unique Sylow 2-subgroup.
- This subgroup is normal in G, hence G is not simple.

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3. Prove that groups of order 48 are not simple.

Solution.
- Let G be a group of order 56. Note that $48 = 2^4 \cdot 3$.
- By the 1st Sylow Theorem, let T be a Sylow 2-subgroup of G.
- Note that $|G : T| = 3$
- Let G act by left multiplication on the left cosets of T.
- This yields a homomorphism $\rho : G \to S_3$.
- $Ker\rho < T < G$. So $Ker\rho \neq G$ and hence it is proper.
- By the 1st isomorphism Theorem, $|G|/|Ker\rho| \Big| |S_3|$, we have $|Ker\rho| > 1$ saying that $Ker\rho$ is nontrivial.
- $Ker\rho$ is a nontrivial proper normal subgroup of G.
- G is not simple.

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4. Let G be the group of two-by-two upper triangular matrices with entries in $\mathbb{Z}_3$:

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a, b, c \in \mathbb{Z}_3, a \neq 0, c \neq 0 \right\}.$$

How many Sylow 3-group does G have? Show that there is a Sylow 2-group isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Solution. There are 2 choices for a, 3 for b, and 2 for c, so $|G| = 12$. By the 1st Sylow theorem, there is at least one Sylow 3-group.

Consider the homomorphism $f : G \to \mathbb{Z}_3^\times \times \mathbb{Z}_3^\times$ given by $f\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = (a, c)$. Its kernel $Kerf = \{\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in \mathbb{Z}_3\}$ has order 3, hence it is a Sylow 3-subgroup. The kernel of a homomorphism must be normal. Any two Sylow 3-subgroups are conjugate, so if one of them is normal then they are all equal. Thus, there is only one Sylow 3-group.

The set of matrices with $b = 0$ is a subgroup of order 4 whence it is a Sylow 2-group. Note that it is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ via the map sending $(i, j) \in \mathbb{Z}_2 \times \mathbb{Z}_2$ to $\begin{pmatrix} (-1)^i & 0 \\ 0 & (-1)^j \end{pmatrix}$.

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5. Let G be a finite group of order 231. Prove that every Sylow 11-Subgroup of G is contained in the center $Z(G)$.

Solution. Note that $237 = 3 \times 7 \times 11$. There is a Sylow 11-Subgroup P of G. We also have that $n_{11}|21$ and $n_{11} \equiv 1 \pmod{11}$. This suggests that $n_{11} = 1$, so there is only one Sylow 11-Subgroup P of G, and hence it is normal in G.

Now we consider the action of G on the normal subgroup P given by conjugation. The action induces the permutation representation homomorphism $\phi : G \to Aut(P)$ where $Aut(P)$ is the automorphism group of P. Here P is a cyclic group of 11, so

$$Aut(P) \simeq \mathbb{Z}_{11}^\times.$$

The first isomorphism theorem gives

$$|G/Ker\phi| \Big| |Aut(P)| = |\mathbb{Z}_{11}^\times| = 10,$$

but $|G| = 237$ implies that $|G/Ker\phi| = 1$. Therefore $G = Ker\phi$. This suggests that for any $g \in G$, $\phi(g) : P \to P$ given by $h \mapsto ghg^{-1}$ is the identity map. Thus we have $ghg^{-1} = h$ for all $g \in G$ and all $h \in P$. It yields that $P < Z(G)$.

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