THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH 3030 Abstract Algebra 2019-20
Tutorial 4
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1. Find the center $Z(G)$ if $G =$

   (a) $S_n (n \geq 3)$.

   (b) D_n.

   (c) $GL(2, \mathbb{R})$.

Solution. (a) $Z(G) = \{\text{Id}\}$. Note that for $n \geq 3$, the only element of S_n commuting with all the elements of the same group is the identity permutation.

(b) Recall that the dihedral group D_n is the group of symmetries of a regular n-gon. This group has $2n$ elements, n of which are rotations about the origin and the other n of which are reflections each with respect to a straight line through the origin. The rotations form a cyclic subgroup, generated by a member σ_0 which is the rotation about the origin by $+\frac{2\pi}{n}$ or $-\frac{2\pi}{n}$ radians, while the reflections form a coset of $\langle\sigma\rangle$. In addition, the elements of D_n satisfy $\sigma^n = r^2 = \text{Id}$ and $r\sigma r = \sigma^{-1}$ for any rotation σ and any reflection r.

- Suppose $Z(D_n)$ contains a reflection r.
  - From the property discussed above, $r\sigma_0 r = \sigma_0^{-1}$.
  - While from the assumption that $r \in Z(D_n)$, $r\sigma_0 = \sigma_0 r$.
  - Hence we have
  $$\sigma_0^{-1} = r\sigma_0 r = \sigma_0 r r \sigma_0.$$
  - That is $\sigma_0^2 = \text{Id}$.
  - But this is impossible since the order of σ_0 is n which is greater than 2.
  - We conclude that $Z(D_n)$ does not contain any element which is a reflection.
- Suppose $Z(D_n)$ contains a rotation $\sigma = \sigma_0^i$ for some $i \in \{0, 1, \ldots, n-1\}$.
  - We have $r\sigma_0^i r = \sigma_0^{-i}$ and $r\sigma_0^i = \sigma_0^i r$.
  - Hence
  $$\sigma_0^{-i} = r\sigma_0^i r = \sigma_0^i r r = \sigma_0^i.$$
  - That is $\sigma_0^{2i} = \text{Id}$.
  - This is possible only if $i = 0$ or $i = \frac{n}{2}$ if n is even.
  - We can also see that σ_0^i indeed commutes with all elements of D_n if i is one of these values.
  - Thus we conclude that
  $$Z(D_n) = \begin{cases} \{\text{Id}\} & \text{if } n \text{ is odd} \\ \{\text{Id}, \sigma_0^{\frac{n}{2}}\} & \text{if} n \text{ is even} \end{cases}.$$

Remark. $\sigma_0^{\frac{n}{2}}$ is the reflection with respect to the origin.

(c)
- Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an element of $Z(GL(2, \mathbb{R}))$.
- Consider the matrix $\begin{pmatrix} \lambda & 0 \\ 0 & \tau \end{pmatrix}$ where λ, τ are nonzero real numbers.
- Then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \lambda & 0 \\ 0 & \tau \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \tau \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$
$$\begin{pmatrix} a\lambda & b\tau \\ c\lambda & d\tau \end{pmatrix} = \begin{pmatrix} a\lambda & b\lambda \\ c\tau & d\tau \end{pmatrix}.$$

- So if we take $\lambda = 1$ and $\tau = 2$, we have $b = 2b$ and $c = 2c$, and hence $b = c = 0$.
- Now consider the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.
- Then

$$\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$$
$$\begin{pmatrix} a & a \\ 0 & d \end{pmatrix} = \begin{pmatrix} a & d \\ 0 & d \end{pmatrix}.$$

- Hence $a = d$.
- Conversely, every element of the form $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ commutes with all 2×2 matrices (even the singular ones).
- We conclude that

$$Z(GL(2, \mathbb{R})) = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} | a \in \mathbb{R} - \{0\} \right\}.$$

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2. Find the commutator subgroup $[G, G]$ if $G =$
   (a) $S_n (n \geq 3)$.
   (b) D_n.
   (c) $A_n (n \geq 3)$.

Solution. (a) We prove that $[S_n, S_n] = A_n$.

- Note first that S_n/A_n is abelian (it is a cyclic group of order 2), and so $[S_n, S_n] < A_n$.

- To show that the inclusion is an equality, it suffices to show that $[S_n, S_n]$ contains all 3-cycles because of Tutorial 1.
- This is true because $(a\ b\ c) = (b\ c)(a\ b)(b\ c)^{-1}(a\ b)^{-1}$.

(b) We prove that $[D_n, D_n] = \langle\sigma^2\rangle$.
Since $rsrs^{-1} = \sigma^{-2}$, every even power of s is in the commutator.
If n is odd, an even power of s is just the same as a power of σ, and since $\langle\sigma\rangle$ is normal and has abelian quotient of order 2, this is exactly the commutator subgroup.
If n is even, then the commutator is $\langle\sigma^2\rangle$, since its quotient is of order 4, which is abelian. Hence $\langle\sigma^2\rangle$ is exactly the commutator subgroup.
Alternately, we can have:

- Note first that the subgroup $\langle\sigma_0\rangle$ has index 2 and the subgroup $D_n \cap A_n$ has index at most 2.
- It follows that they are normal in D_n and the quotient groups of D_n defined by them are abelian.
- Hence $[D_n, D_n] < \langle\sigma_0\rangle \cap (D_n \cap A_n) = \langle\sigma_0\rangle \cap A_n$.
- We claim that the right member is $\langle\sigma_0^2\rangle$.
- Since σ_0^2 is an even permutation, $\langle\sigma_0^2\rangle \subseteq \langle\sigma_0\rangle \cap A_n$.
- On the other hand, if σ_0^i is an even permutation for some i, and if i is odd, then σ_0 is an even permutation, and hence n is odd.
- But if n is odd, then $\langle\sigma_0\rangle = \langle\sigma_0^2\rangle$ so that $\sigma_0^i \in \langle\sigma_0^2\rangle$. Our claim is proved.
- Now we do have $[D_n, D_n] = \langle\sigma_0^2\rangle$.
- It follows from the observation that $r\sigma_0^{-1}r = \sigma_0 \Longrightarrow \sigma_0^2 = \sigma_0 r\sigma_0^{-1}r^{-1}$.

(c)
- $[A_3, A_3] = \{Id\}$ since A_3 is abelian.
- Note that Klevin 4 group V can be identified as $\{Id, (1,2)(3,4), (1,3)(2,4), (1,4)(2,3)\}$. V is a normal subgroup of A_4. Since A_4/V is abelian, $[A_4, A_4] \subset V$. $[A_4, A_4]$ is not trivial, as A_4 nonabelian. The order of $[A_4, A_4]$ must be 2 or 4, so it contains $(a,b)(c,d)$. $[A_4, A_4]$ is normal in A_4, so $(a,b)(c,d)$ is conjugate to every term of $(1,2)(3,4), (1,3)(2,4), (1,4)(2,3)$. Note that $(1,2)(3,4), (1,3)(2,4), (1,4)(2,3)$ are pairwisely conjugate to each other in A_4. In fact we can find some (not all) relations such as $(1,3,2)(1,3)(2,4)(1,2,3) = (1,4)(2,3)$ and $(1,2,3)(1,3)(2,4)(1,3,2) = (1,2)(3,4)$. Then $[A_4, A_4] = \{Id, (1,2)(3,4), (1,3)(2,4), (1,4)(2,3)\}$.
- We can generate all the 3-cycles in A_n for $n \geq 5$. For example, we can consider $a = (1,3,2)$ and $b = (2,3)(4,5)$. Then by direct checking $[a,b] = aba^{-1}b^{-1} = (1,2,3)$. This suggests that $[A_n, A_n] = A_n$ for $n \geq 5$.

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3. Let H, K be two normal subgroup of G such that G/H and G/K are both abelian. Show that $G/(H \cap K)$ is abelian.

Solution. We must have $[G,G] \subset H$ and $[G,G] \subset K$. In particular $[G,G] \subset H \cap K$. Then we are done because of the fact that the subgroup N is normal in G and G/N is abelian if and only if $[G,G] \subset N$.

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4. Show that if $N \triangleleft G$ and $G = Z(G)N$, then $[G, G] < N$.

Solution.
- It suffices to show that the quotient group G/N is abelian.
- Consider the projection map $\pi : G \to G/N$ and consider the image $\pi(Z(G))$ of $Z(G)$ under π.
- Let $x \in G/N$ be an element, then $x = \pi(g)$ for some $g \in G$.
- From $G = Z(G)N$, there are $h \in Z(G)$ and $k \in N$ such that $g = hk$.
- It follows that $x = \pi(hk) = \pi(h) \cdot e = \pi(h) \in \pi(Z(G))$.
- This shows that $G/N = \pi(Z(G))$, and hence G/N is abelian.

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5. Let H be any group with $|H| = h$, where h is odd. Show that $A_4 \times H$ has no subgroups of order $6h$.

Solution. Suppose to the contrary that $A_4 \times H$ has a subgroup K with $|K| = 6h$; then K has index 2 in $A_4 \times H$, so that K is a normal subgroup of $A_4 \times H$ and $|(A_4 \times H)/K| = 2$. Hence, $(A_4 \times H)/K$ is Abelian, so that $[A_4 \times H, A_4 \times H]$ is a subgroup of K. Also note that $[A_4 \times H, A_4 \times H] = [A_4, A_4] \times [H, H]$. It follows that $|[A_4, A_4]|$ divides $|K|$. Now $[A_4, A_4] = \{Id, (1,2)(3,4), (1,3)(2,4), (1,4)(2,3)\}$, so that 4 must divide $|K| = 6h$; this is not possible, since h is odd, and this completes our proof.

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