Math 2230B, Complex Variables with Applications

1. With the aid of identity (see Sec. 37)
$$\cos z = -\sin(z - \frac{\pi}{2}),$$
expand $\cos z$ into a Taylor series about the point $z_0 = \pi/2$.

2. Use representation (3), Sec. 64, for $\sin z$ to write the Maclaurin series for the function
$$f(z) = \sin(z^2),$$
and point out how it follows that
$$f^{(4n)}(0) = 0 \quad \text{and} \quad f^{(2n+1)}(0) = 0 \quad (n = 0, 1, 2, ...).$$

3. Derive the expansions

   (a) $\frac{\sinh z}{z^2} = \frac{1}{z} + \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+3)!} \quad (0 < |z| < \infty)$

   (b) $\frac{\sin(z^2)}{z^4} = \frac{1}{z^2} - \frac{z^2}{3!} + \frac{z^6}{5!} - \frac{z^{10}}{7!} + \cdots \quad (0 < |z| < \infty)$.

4. Show that when $0 < |z| < 4$,
$$\frac{1}{4z - z^2} = \frac{1}{4z} + \Sigma_{n=0}^{\infty} \frac{z^n}{4^{n+2}}.$$

5. Find the Laurent series that represents the function
$$f(z) = z^2 \sin\left(\frac{1}{z^2}\right)$$
in the domain $0 < |z| < \infty$.

6. Find a representation for the function
$$f(z) = \frac{1}{1+z} = \frac{1}{z} \cdot \frac{1}{1 + (1/z)}$$
in negative powers of z that is valid when $1 < |z| < \infty$.

7. The function
$$f(z) = \frac{-1}{(z-1)(z-2)} = \frac{1}{z-1} - \frac{1}{z-2}$$
which has the two singular points $z = 1$ and $z = 2$, is analytic in the domains
$$D_1 : |z| < 1, D_2 : 1 < |z| < 2, D_3 : 2 < |z| < \infty.$$
Find the series representation in powers of z for $f(z)$ in each of these domains.

8. Show that when $0 < |z-1| < 2$,

$$\frac{z}{(z-1)(z-3)} = -3\sum_{n=0}^{\infty}\frac{(z-1)^n}{2^{n+2}} - \frac{1}{2(z-1)}.$$

9. (a) Let a denote a real number, where $-1 < a < 1$, and derive the Laurent series representation

$$\frac{a}{z-a} = \sum_{n=1}^{\infty}\frac{a^n}{z^n} \quad (|a| < |z| < \infty)$$

.

(b) After writing $z = e^{i\theta}$ in the equation obtained in part (a), equate real parts and then imaginary parts on each side of the result to derive the summation formulas

$$\sum_{n=1}^{\infty} a^n \cos n\theta = \frac{a\cos\theta - a^2}{1 - 2a\cos\theta + a^2} \quad \text{and} \quad \sum_{n=1}^{\infty} a^n \sin n\theta = \frac{a\sin\theta}{1 - 2a\cos\theta + a^2},$$

where $-1 < a < 1$.

10. (a) Let z be any complex number, and let C denote the unit circle

$$w = e^{i\phi} \quad (-\pi \le \phi \le \pi)$$

in the ω plane. Then use that contour in expansion (5), Sec. 66, for the coefficients in a Laurent series, adapted to such series about the origin in the ω plane, to show that

$$\exp\left[\frac{z}{2}\left(w - \frac{1}{w}\right)\right] = \sum_{n=-\infty}^{\infty} J_n(z)w^n \quad (0 < |w| < \infty)$$

where

$$J_n(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \exp[-i(n\phi - z\sin\phi)]d\phi \quad (n = 0, \pm1, \pm2, \ldots).$$

(b) With the aid of Exercise 5, Sec. 42, regarding certain definite integrals of even and odd complex-valued functions of a real variable, show that the coefficients in part (a) here can be written

$$J_n(z) = \frac{1}{\pi}\int_{0}^{\pi} \cos(n\phi - z\sin\phi)d\phi \quad (n = 0, \pm1, \pm2, \ldots).$$

11. (a) Let f(z) denote a function which is analytic in some annular domain about the origin that includes the unit circle $z = e^{i\phi}(-\pi \leq \phi \leq \pi)$. By taking that circle as the path of integration i expression (2) and (3), Sec. 66, for the coefficient a_n and b_n in a Laurent series in power of z, show that

$$f(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\phi})d\phi + \frac{1}{2\pi}\Sigma_{n=1}^{\infty}\int_{-\pi}^{\pi} f(e^{i\phi})[(\frac{z}{e^{i\phi}})^n + (\frac{e^{i\phi}}{z})^n]d\phi$$

when z is any point in the annular domain.

(b) Write $u(\theta) = Re[f(e^{i\theta})]$ and show how it follows from the expansion in part (a) that

$$f(\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(\phi)d\phi + \frac{1}{\pi}\Sigma_{n=1}^{\infty}\int_{-\pi}^{\pi} u(\phi)\cos[n(\theta - \phi)]d\phi$$

This is one form of the **Fourier series** expansion of the real-valued function $u(\theta)$ on the interval $-\pi \leq \theta \leq \pi$. The restriction on $u(\theta)$ is more severe than is necessary in order for it to be represented by a Fourier series.