Math 2230B, Complex Variables with Applications

1. Use an antiderivative to show that for every contour C extending from a point z_1 to a point z_2,

$$\int_C z^n dz = \frac{1}{n+1}\left(z_2^{n+1} - z_1^{n+1}\right) \quad (n = 0, 1, 2, \ldots)$$

.

2. By finding an antiderivative, evaluate each of these integrals, where the path is any contour between the indicated limits of integration:

$$\text{(a)} \int_0^{1+i} z^2 dz \quad \text{(b)} \int_0^{\pi+2i} \cos\left(\frac{z}{2}\right) dz; \quad \text{(c)} \int_1^3 (z-2)^3 dz$$

3. Use the theorem in Sec.48 to show that

$$\int_{C_0} (z - z_0)^{n-1}\, dz = 0 \quad (n = \pm 1, \pm 2, \ldots)$$

when C_0 is any closed contour which does not pass through the point z_0. (Compare with Exercise 13, Sec. 46.)

4. Show that

$$\int_{-1}^{1} z^i dz = \frac{1 + e^{-\pi}}{2}(1 - i),$$

where the integrand denotes the principal branch

$$z^i = \exp(i\text{Log}z) \quad (|z| > 0, -\pi < \text{Arg}\, z < \pi)$$

of z^i and where the path of integration is any contour from $z = -1$ to $z = 1$ that, except for its end points, lies above the real axis. (Compare with Exercise 6, Sec. 46.) *Suggestion:* Use an antiderivative of the branch

$$z^i = \exp(i \log z) \quad \left(|z| > 0, -\frac{\pi}{2} < \arg z < \frac{3\pi}{2}\right)$$

of the same power function.

5. Let C_1 denote the positively oriented boundary of the square whose sides lie along the lines $x = \pm 1$, $y = \pm 1$ and let C_2 be the positively oriented circle $|z| = 4$(Fig.63). With the aid of the corollary in Sec.53, point out why

$$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz$$

when

$$\text{(a) } f(z) = \frac{1}{3z^2+1}; \qquad \text{(b) } f(z) = \frac{z+2}{\sin(z/2)}; \qquad \text{(c) } f(z) = \frac{z}{1-e^z}.$$

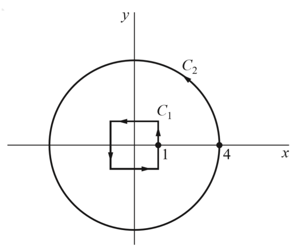

FIGURE 63

6. If C_0 denotes a positively oriented circle $|z - z_0| = R$, then

$$\int_{C_0} (z-z_0)^{n-1}\,dz = \begin{cases} 0 & \text{when } n = \pm1, \pm2, \ldots \\ 2\pi i & \text{when } n = 0, \end{cases}$$

according to Exercise 13, Sec. 46. Use that result and the corollary in Sec. 53 to show that if C is the boundary of the rectangle $0 \le x \le 3$, $0 \le y \le 2$, described in the positive sense, then

$$\int_C (z-2-i)^{n-1} dz = \begin{cases} 0 & \text{when } n = \pm1, \pm2, \ldots \\ 2\pi i & \text{when } n = 0. \end{cases}$$

7. Use the following method to derive the integration formula

$$\int_0^\infty e^{-x^2} \cos 2bx dx = \frac{\sqrt{\pi}}{2} e^{-b^2} \quad (b > 0).$$

(a) Show that the sum of the integral of e^{-z^2} along the lower and upper horizontal legs of the rectangular path in Fig. 64 can be written

$$2\int_0^a e^{-x^2} dx - 2e^{b^2} \int_0^a e^{-x^2} \cos 2bx dx$$

and that the sum of the integrals along the vertical legs on the right and left can be written

$$ie^{-a^2} \int_0^b e^{y^2} e^{-i2ay} dy - ie^{-a^2} \int_0^b e^{y^2} e^{i2ay} dy.$$

Thus, with the aid of the Cauchy-Goursat theorem show that

$$\int_0^a e^{-x^2} \cos 2bx dx = e^{-b^2} \int_0^a e^{-x^2} dx + e^{-(a^2+b^2)} \int_0^b e^{y^2} \sin 2ay dy.$$

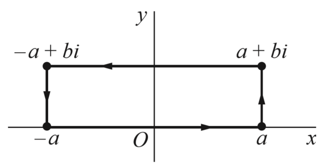

FIGURE 64

(b) By accepting the fact that

$$\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$$

and observing that

$$\left|\int_0^b e^{y^2}\sin 2ay\,dy\right| \le \int_0^b e^{y^2}\,dy,$$

obtain the desired integration formula by letting a tend to infinity in the equation at the end of part (a).