Math 2230A, Complex Variables with Applications

1. Derive the expansions

   (a) $\frac{\sinh z}{z^2} = \frac{1}{z} + \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+3)!} \quad (0 < |z| < \infty)$

   (b) $\frac{\sin(z^2)}{z^4} = \frac{1}{z^2} - \frac{z^2}{3!} + \frac{z^6}{5!} - \frac{z^{10}}{7!} + \cdots \quad (0 < |z| < \infty).$

2. Show that when $0 < |z| < 4$,

$$\frac{1}{4z - z^2} = \frac{1}{4z} + \sum_{n=0}^{\infty} \frac{z^n}{4^{n+2}}.$$

3. Find the Laurent series that represents the function

$$f(z) = z^2 \sin\left(\frac{1}{z^2}\right)$$

   in the domain $0 < |z| < \infty$.

4. Find a representation for the function

$$f(z) = \frac{1}{1+z} = \frac{1}{z} \cdot \frac{1}{1 + (1/z)}$$

   in negative powers of z that is valid when $1 < |z| < \infty$.

5. The function

$$f(z) = \frac{-1}{(z-1)(z-2)} = \frac{1}{z-1} - \frac{1}{z-2}$$

   which has the two singular points $z = 1$ and $z = 2$, is analytic in the domains

$$D_1 : |z| < 1, D_2 : 1 < |z| < 2, D_3 : 2 < |z| < \infty.$$

   Find the series representation in powers of z for $f(z)$ in each of these domains.

6. Show that when $0 < |z - 1| < 2$,

$$\frac{z}{(z-1)(z-3)} = -3 \sum_{n=0}^{\infty} \frac{(z-1)^n}{2^{n+2}} - \frac{1}{2(z-1)}.$$

7. (a) Let a denote a real number, where $-1 < a < 1$, and derive the Laurent series representation

$$\frac{a}{z-a} = \sum_{n=1}^{\infty} \frac{a^n}{z^n} \quad (|a| < |z| < \infty)$$

.

(b) After writing $z = e^{i\theta}$ in the equation obtained in part (a), equate real parts and then imaginary parts on each side of the result to derive the summation formulas

$$\sum_{n=1}^{\infty} a^n \cos n\theta = \frac{a\cos\theta - a^2}{1 - 2a\cos\theta + a^2} \quad \text{and} \quad \sum_{n=1}^{\infty} a^n \sin n\theta = \frac{a\sin\theta}{1 - 2a\cos\theta + a^2},$$

where $-1 < a < 1$.

8. (a) Let z be any complex number, and let C denote the unit circle

$$w = e^{i\phi} \quad (-\pi \le \phi \le \pi)$$

in the ω plane. Then use that contour in expansion (5), Sec. 66, for the coefficients in a Laurent series, adapted to such series about the origin in the ω plane, to show that

$$\exp\left[\frac{z}{2}\left(w - \frac{1}{w}\right)\right] = \sum_{n=-\infty}^{\infty} J_n(z) w^n \quad (0 < |w| < \infty)$$

where

$$J_n(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp[-i(n\phi - z\sin\phi)] d\phi \quad (n = 0, \pm 1, \pm 2, \ldots).$$

(b) With the aid of Exercise 5, Sec. 42, regarding certain definite integrals of even and odd complex-valued functions of a real variable, show that the coefficients in part (a) here can be written

$$J_n(z) = \frac{1}{\pi} \int_{0}^{\pi} \cos(n\phi - z\sin\phi) d\phi \quad (n = 0, \pm 1, \pm 2, \ldots).$$

9. In each case, write the principal part of the function at its isolated singular point and determine whether that point is a removable singular point, an essential singular point, or a pole:

$$(a) z\exp\left(\frac{1}{z}\right); \quad \text{(b)}\ \frac{z^2}{1+z}; \quad (c)\frac{\sin z}{z}; \quad (d)\frac{\cos z}{z}; \quad (e)\frac{1}{(2-z)^3}.$$

10. Show that the singular point of each of the following functions is a pole. Determine the order m of that pole and the corresponding residue B.

$$\text{(a)}\ \frac{1-\cosh z}{z^3};\qquad \text{(b)}\ \frac{1-\exp(2z)}{z^4};\quad (c)\frac{\exp(2z)}{(z-1)^2}.$$