Math 2230A, Complex Variables with Applications

1. Let C_1 denote the positively oriented boundary of the square whose sides lie along the lines $x = \pm 1$, $y = \pm 1$ and let C_2 be the positively oriented circle $|z| = 4$(Fig.63). With the aid of the corollary in Sec.53, point out why

$$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz$$

when

$$\text{(a) } f(z) = \frac{1}{3z^2+1}; \qquad \text{(b) } f(z) = \frac{z+2}{\sin(z/2)}; \qquad \text{(c) } f(z) = \frac{z}{1-e^z}.$$

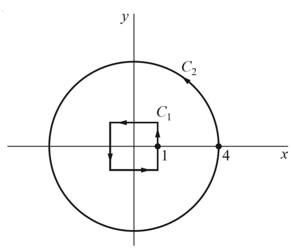

FIGURE 63

2. If C_0 denotes a positively oriented circle $|z - z_0| = R$, then

$$\int_{C_0} (z - z_0)^{n-1}\, dz = \begin{cases} 0 & \text{when } n = \pm 1, \pm 2, \ldots \\ 2\pi i & \text{when } n = 0, \end{cases}$$

according to Exercise 13, Sec. 46. Use that result and the corollary in Sec. 53 to show that if C is the boundary of the rectangle $0 \le x \le 3$, $0 \le y \le 2$, described in the positive sense, then

$$\int_C (z - 2 - i)^{n-1} dz = \begin{cases} 0 & \text{when } n = \pm 1, \pm 2, \ldots \\ 2\pi i & \text{when } n = 0. \end{cases}$$

3. Use the following method to derive the integration formula

$$\int_0^\infty e^{-x^2} \cos 2bx dx = \frac{\sqrt{\pi}}{2} e^{-b^2} \quad (b > 0).$$

   (a) Show that the sum of the integral of e^{-z^2} along the lower and upper horizontal legs of the rectangular path in Fig. 64 can be written

$$2\int_0^a e^{-x^2} dx - 2e^{b^2} \int_0^a e^{-x^2} \cos 2bx dx$$

and that the sum of the integrals along the vertical legs on the right and left can be written

$$ie^{-a^2}\int_0^b e^{y^2}e^{-i2ay}dy - ie^{-a^2}\int_0^b e^{y^2}e^{i2ay}dy.$$

Thus, with the aid of the Cauchy-Goursat theorem show that

$$\int_0^a e^{-x^2}\cos 2bxdx = e^{-b^2}\int_0^a e^{-x^2}dx + e^{-\left(a^2+b^2\right)}\int_0^b e^{y^2}\sin 2aydy.$$

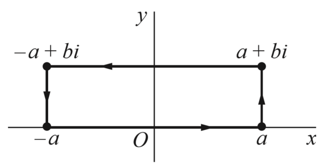

FIGURE 64

(b) By accepting the fact that

$$\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$$

and observing that

$$\left|\int_0^b e^{y^2}\sin 2aydy\right| \leq \int_0^b e^{y^2}dy,$$

obtain the desired integration formula by letting a tend to infinity in the equation at the end of part (a).

4. According to Exercise 6, Sec.43, the path C_1 from the origin to the point $z = 1$ along the graph of the function defined by means of the equations

$$y(x) = \begin{cases} x^3\sin(\pi/x) & \text{when } 0 < x \leq 1, \\ 0 & \text{when } x = 0. \end{cases}$$

is a smooth arc that intersects the real axis an infinite number of times. Let C_2 denote the line segment along the real axis from $z = 1$ back to the origin, and let C_3 denote any smooth arc from the origin to $z = 1$ that does not intersect itself and has only its end points in common with the arcs C_1 and C_2 (Fig. 65). Apply the Cauchy-Goursat theorem to show that if a function f is entire, then

$$\int_{C_1} f(z)dz = \int_{C_3} f(z)dz \quad \text{and} \quad \int_{C_2} f(z)dz = -\int_{C_3} f(z)dz.$$

Conclude that even though the closed contour $C = C_1 + C_2$ intersects itself an infinite number of times,

$$\int_C f(z)dz = 0.$$

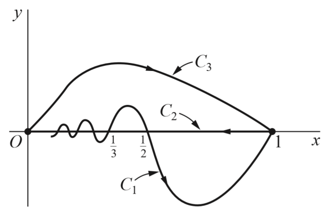

FIGURE 65

5. Let C denote the positively oriented boundary of the half disk $0 \leq r \leq 1$, $0 \leq \theta \leq \pi$, and let f(z) be a continuous function defined on that half disk by writing $F(0) = 0$ and using the branch

$$f(z) = \sqrt{r}e^{i\theta/2} \quad \left(r > 0, -\frac{\pi}{2} < \theta < \frac{3\pi}{2}\right)$$

of the multiple-valued function $z^{1/2}$. Show that

$$\int_C f(z)dz = 0$$

by evaluating separately the integrals of f(z) over the semicircle and the two radii which make up C. Why does the Cauchy-Goursat theorem does not apply here.

6. Let C denote the positively oriented boundary of the square whose slides lie along the lines $x = \pm 2$ and $y = \pm 2$. Evaluate each of these integrals:

$$\text{(a)} \int_C \frac{e^{-z}dz}{z-(\pi i/2)}; \quad \text{(b)} \int_C \frac{\cos z}{z(z^2+8)}dz; \quad (c) \int_C \frac{zdz}{2z+1}$$
$$\text{(d)} \int_C \frac{\cosh z}{z^4}dz; \quad \text{(e)} \int_C \frac{\tan(z/2)}{(z-x_0)^2}dz \quad (-2 < x_0 < 2)$$

7. Find the value of the integral of g(z) around the circle $|z - i| = 2$ in the positive sense when

$$\text{(a)}\ g(z) = \frac{1}{z^2+4}; \quad \text{(b)}\ g(z) = \frac{1}{(z^2+4)^2}.$$

8. Let C be the circle $|z| = 3$, described in the positive sense. Show that if
$$g(z) = \int_C \frac{2s^2 - s - 2}{s - z} ds \quad (|z| \neq 3),$$
then $g(2) = 8\pi i$. What is the value of $g(z)$ when $|z| > 3$?

9. Let C be any simple closed contour, described in the positive sense in the z plane, and write
$$g(z) = \int_C \frac{s^3 + 2s}{(s - z)^3} ds.$$
Show that $g(z) = 6\pi i z$ when z is inside C and that $g(z) = 0$ when z is outside.

10. Let f be an entire function such that $|f(z)| \leq A|z|$ for all z, where A is a fixed positive number. Show that $f(z) = a_1 z$, where a_1 is a complex constant.
*Suggestion:*Use Cauchy's inequality (Sec. 57) to show that the second derivative $f''(z)$ is zero everywhere in the plane. Note that the constant M_R in Cauchy's inequality is less than or equal to $A(|z_0| + R)$.

11. Let R region $0 \leq x \leq \pi$, $0 \leq y \leq 1$ (Fig. 72). Show that the modulus of the entire function $f(z) = \sin z$ has a maximum value in R at the boundary point $z = (\pi/2) + i$.
Suggestion: Write $|f(z)|^2 = \sin^2 x + \sinh^2 y$(see Sec.37) and locate points in R at which $\sin^2 x$ and $\sinh^2 y$ are the largest.

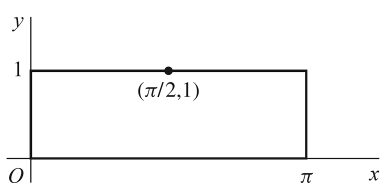

FIGURE 72

12. Let f be the function $f(z) = e^z$ and R the rectangular region $0 \leq x \leq 1$, $0 \leq y \leq \pi$. Illustrate results in Sec. 59 and exercise 5 by finding points in R where the component function $u(x, y) = \text{Re}[f(z)]$ reaches its maximum and minimum values.