THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH 2070A Algebraic Structures 2019-20
Tutorial 6
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Problems:

1. Mark each of the following statements “True” (meaning that it is a true statement) or “False” (meaning that there are counterexamples to the statement or disprove to the statement). No reasoning is required.

   (a) Any group of order 6 is cyclic.

   (b) Any group of order 6 is abelian.

   (c) It is possible that a group of order 6 has an element of order 4.

   (d) It is not possible to have a nontrivial homomorphism of a finite group to an infinite group.

   (e) There is a nontrivial homomorphism form $\mathbb{Z}_{2010}$ to $\mathbb{Z}$.

   (f) There is a nontrivial homomorphism form S_4 to S_3.

Solution. (a) F. Consider S_3.

(b) F. Consider S_3.

(c) F. Because $|a|$ divides $|G|$ for any finite group G and $a \in G$.

(d) F. Consider $\phi : \mathbb{Z}_2 \to \mathbb{Z}_2 \times \mathbb{Z}$ defined by $\phi(n) = (n, 0)$.

(e) F. There is no nontrivial finite subgroup in $\mathbb{Z}$.

(f) T. Let $\phi(\sigma) = (1, 2)$ for each odd permutation $\sigma \in S_4$, and let $\phi(\sigma)$ be the identity permutation for each even $\sigma \in S_4$.

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2. Determine whether the given subset is a subgroup. If it is a subgroup, prove it. If it is not a subgroup, explain why.

   The set of matrices having trace 0 inside $GL_2(\mathbb{R})$ equipped with matrix multiplication. (Trace of matrix is the sum of diagonals.)

   Solution. It is not closed. Consider $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^2 = I_2$.

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3. Write down all the left cosets of $\langle 3 \rangle$ in $\mathbb{Z}_{12}$.

   Solution. $\langle 3 \rangle = \{0, 3, 6, 9\}$, $1 + \langle 3 \rangle = \{1, 4, 7, 10\}$, $2 + \langle 3 \rangle = \{2, 5, 8, 11\}$.

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4. Find all of the homomorphisms $\phi : \mathbb{Z}_{15} \to \mathbb{Z}_6$.

   Solution. Let ϕ be a homomorphism from $\mathbb{Z}_{15}$ to $\mathbb{Z}_6$. Let $\phi(1) = a$. Notice that

   $$0 = \phi(0) = \phi(15) = 15\phi(1) = 15a.$$

   We then get 6 divides $15a$. So a can be $0, 2, 4$.

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5. Let $\mathbb{Z}_{18}^{\times}$ be the group of all positive integers less than 18 and relative prime to 18, with the group operation given by the multiplication modulo 18. Show that $\mathbb{Z}_{18}^{\times}$ is cyclic.

   Solution. First we have $\mathbb{Z}_{18}^{\times} = \{1, 5, 7, 11, 13, 17\}$. The order of 5 is 6 which is the order of $|\mathbb{Z}_{18}^{\times}|$, as $5^2 \equiv 7 \pmod{18}$ and $5^3 \equiv 17 \pmod{18}$.

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6. Under the addition and multiplication as the operations in $\mathbb{C}$, determine whether the set of imaginary complex numbers is a ring.

   Solution. No. It is not closed under multiplication. ◀

Optional Part

1. Let H and K be subgroups of a group G. Define

$$HK = \{hk | h \in H, k \in K\}.$$

   (a) Show that HK is a subgroup of G if and only if $HK = KH$.

   (b) Give an example a group G and two subgroups H and K such that HK is not a subgroup of G.

   (c) Let $V_4 = \{Id, a, b, c\}$ where $a = (1,2)(3,4)$, $b = (1,3)(2,4)$ and $c = (1,4)(2,3)$. Let $H_1 = \langle a \rangle$, $H_2 = \langle b \rangle$ and $H_3 = \langle a \rangle$. Show that $H_i \cap H_j = \{Id\}$ for all $i \neq j$ and $V_4 = H_i H_j$ for all i and j.

Solution. (a) Suppose HK is a subgroup of G. Let $kh \in KH$ where $h \in H$ and $k \in K$. Then $h = he \in HK$ and $k = ek \in HK$ and since HK is closed under products, we deduce $kh \in HK$. Thus we have $KH \subset HK$.

Also let $hk \in HK$ where $h \in H$ and $k \in K$. We have $(hk)^{-1} \in HK$, so $(hk)^{-1} = xy$ where $x \in H$ and $y \in K$. Then $hk = (xy)^{-1} = y^{-1}x^{-1} \in KH$ (as $x^{-1} \in H$ and $y^{-1} \in K$). This shows that $HK \subset KH$.

Conversely suppose $HK = KH$. Let $a, b \in HK$; say $a = h_1k_1$ and $b = h_2k_2$ where $h_1, h_2 \in H$ and $k_1, k_2 \in K$. Then $k_1h_2 \in KH = HK$; say $k_1h_2 = hk$ where $h \in H$ and $k \in K$. Now $ab = h_1k_1h_2k_2 = h_1hkk_2 \in HK$ since $h_1h \in H$ and $kk_2 \in K$. Also $a^{-1} = (h_1k_1)^{-1} = k_1^{-1}h_1^{-1} \in KH = HK$. Hence HK is closed under products and inverses, so it is a subgroup of G.

(b) Take $G = S_3$, $H = \langle(1,2)\rangle$ and $K = \langle(2,3)\rangle$. Then H and K are subgroups of G (each containing two elements) and

$$HK = \{Id, (1,2), (2,3), (1,2,3)\}$$

a set of size 4. Therefore HK is not a subgroup of G (by Lagrange's Theorem) since 4 does not divide 6.

(c) Since a, b and c are permutations of order 2, we have $|H_i| = 2$ for all i. Since these subgroups are distinct, clearly $H_i \cap H_j$ is a proper subgroup of H_i for all i and j, so $H_i \cap H_j = \{Id\}$ by Lagrange's Theorem.

Since V_4 is an abelian group, H_iH_j is a subgroup of V_4 (by Question (a)) while it contains at least three elements (namely those in $H_i \cup H_j$). Hence $|H_iH_j| = 4$ by Lagrange's Theorem and we deduce $V_4 = H_iH_j$.

[Alternatively $H_i \cap H_j = \{Id\}$ can be checked directly and that $V_4 = H_iH_j$ can be done by simply calculating all the elements in H_iH_j directly.]

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