THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH 2070A Algebraic Structures 2019-20
Tutorial 5
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Problems:

1. Which of the following maps are homomorphisms? If the map is a homomorphism, what is the kernel?

   (a) $\phi_1 : \mathbb{R}^\times \to GL_2(\mathbb{R})$ defined by $\phi_1(a) = \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}$.

   (b) $\phi_2 : \mathbb{R} \to GL_2(\mathbb{R})$ defined by $\phi_2(a) = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}$.

   (c) $\phi_3 : GL_2(\mathbb{R}) \to \mathbb{R}$ defined by $\phi_3\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = a + d$.

Solution. (a) It is a homomorphism as

$$\forall a, b \in \mathbb{R}^\times\ \phi_1(a \cdot b) = \begin{pmatrix} 1 & 0 \\ 0 & ab \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix} = \phi_1(a)\phi_1(b).$$

The kernel is $\{1\}$.

(b) It is a homomorphism since

$$\forall a, b \in \mathbb{R}\ \phi_2(a + b) = \begin{pmatrix} 1 & 0 \\ a+b & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} = \phi_2(a)\phi_2(b).$$

The kernel is $\{0\}$.

(c) It is not a homomorphism because

$$\phi_3\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = 2 \neq 0$$

◀

2. Show that the inverse map of a bijective group homomorphism is still a group homomorphism.

Solution. Let G and H be groups and let $\varphi : G \to H$ be a bijective group homomorphism. Since φ is bijective, there exists a map $\psi : H \to G$ such that

$$\psi \circ \varphi = id_G \quad \text{and} \quad \varphi \circ \psi = id_H.$$

To prove $\psi : H \to G$ is a group homomorphism, we need to show that

$$\forall a, b \in H \ \psi(ab) = \psi(a)\psi(b).$$

Actually we have for any $a, b \in H$,

$$\varphi(\psi(ab)) = ab = \varphi(\psi(a))\varphi(\psi(b)) \Rightarrow \psi(ab) = \psi(a)\psi(b)$$

where in the last step we have used that φ is injective. Then we are done.

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3. Let $f : (\mathbb{R}, +) \to (\mathbb{R}_{>0}, \times), x \mapsto e^x$. Show that f is a (group) isomorphism. Hence or otherwise, show that $\ln : (\mathbb{R}_{>0}, \times) \to (\mathbb{R}, +)$ is an isomorphism.

Solution. $f(x) = e^x$ is clearly a well-defined function from $\mathbb{R}$ to $\mathbb{R}_{>0}$ and is bijective. By Qn 2, it suffices to show f is a homomorphism.

Let $x, y \in \mathbb{R}$. Then $f(x + y) = e^{x+y} = e^x \times e^y = f(x)f(y)$.

Thus f is a homomorphism and hence an isomorphism.

Since $\ln$ is the inverse function of e^x, it is an isomorphism by Qn 2.

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4. Let $\mathbb{Z}_8^\times$ be the group of all positive integers less than 8 and relative prime to 8. Show that $\mathbb{Z}_8^\times \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ (as groups).

Solution. Define $g : \mathbb{Z}_8^\times \to \mathbb{Z}_2 \times \mathbb{Z}_2$ by

$$g(1) = (0, 0), \quad g(3) = (0, 1), \quad g(5) = (1, 0), \quad g(7) = (1, 1).$$

Check (i) g is a bijection, (ii) g is a homomorphism.

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5. Let G be a group. Is the map $\phi : G \to G$ by $\phi(g) = g^2$ a group homomorphism?

Solution. No. Take $G = S_3$.

$g((1, 2, 3)(1, 2)) = g(1, 3) = (1, 3)^2 = Id \neq (1, 3, 2) = (1, 2, 3)^2(1, 2)^2 = g(1, 2, 3)g(1, 2)$.

Remark: This is true when G is abelian. ◀

6. Let G be a group. Is the map $\phi : G \to G$ by $\phi(g) = g^{-1}$ a group homomorphism?

Solution. No. Take $G = S_3$. Note that

$$g((1,2,3)(1,2)) = g(1,3) = (1,3)^{-1} = (1,3)$$

but

$$g(1,2,3)g(1,2) = (1,2,3)^{-1}(1,2)^{-1} = (1,3,2)(1,2) = (2,3).$$

Remark: This is true when G is abelian. ◀

Optional Part

1. Show that every finite nonabelian group has a nontrivial automorphism (a group homomorphism from a group to itself).

Solution. Let G be a nonabelian group. There exist $x, y \in G$ such that $xy \neq yx$. Consider the map $T_x : G \to G$ by $T_x(g) = xgx^{-1}$. This map is a group homomorphism and bijective, and also its inverse is a group homomorphism. However this map is nontrivial for otherwise $T_x(y) = y$ implying $xy = yx$, a contradiction as $xy \neq yx$.

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2. Let G be additive group, and u, v are two homomorphisms from G to G. Show that $f : G \to G$, $f = Id_G - v \circ u$ is onto if $h : G \to G$, $h = Id_G - u \circ v$ is onto.

Solution. Note that $f \circ v = v \circ h$ (work it out yourself). For any $y \in G$, there is $x \in G$ such that $-u(y) = h(x)$ since h is onto. For any $y \in G$, we have

$$f(y + v(x)) = f(y) + f \circ v(x) = y - v(u(y)) + v(h(x)) = y + v(-u(y) + h(x)) = y.$$

Then we are done.

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