THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH 2070A Algebraic Structures 2019-20
Homework 4
Due Date: 10th October 2019

Compulsory Part

1. Write down all the cosets of the following subgroups
   (a) $4\mathbb{Z} < \mathbb{Z}$.
   (b) $\langle 4 \rangle < \mathbb{Z}_{12}$.

2. Find a cyclic subgroup of order 4 in S_4, and then give a list of its left coset representatives in S_4.

   (An element g in a group G is a called a **representative** of a left coset S of a subgroup H of G if $S = gH$. Note that g is a representative of S if and only if $g \in S$.)

3. Let H be a subgroup of index 2 in a group G. Show that every left coset of H is also a right coset of H.

4. Let G be a group of order pq, where p and q are prime numbers. Show that every proper subgroup of G is cyclic.

Optional Part

1. Write down all the cosets of the following subgroups
   (a) $4\mathbb{Z} < 2\mathbb{Z}$.
   (b) $\langle 2 \rangle < \mathbb{Z}_{12}$.
   (c) $\langle s \rangle < D_n$ where s is any reflection.

2. Recall the definition of the **quaternion group**:

$$Q = \{\pm 1, \pm i, \pm j, \pm k\},$$

   where the group operation is written multiplicatively,

$$(-1)^2 = 1, i^2 = j^2 = k^2 = ijk = -1,$$

   the symbol 1 denotes the identity element, and -1 commutes with every element of the group.

   Consider the cyclic subgroup $H = \langle i \rangle$ of Q. Find $[Q : H]$, and give a list of representatives of the left cosets of H in Q.

3. Consider the dihedral group $D_6 = \{r_0, r_1, \ldots, r_5, s_1, s_2, \ldots, s_6\}$, where r_0 is the identity element, each r_k corresponds to the anticlockwise rotation by the angle of $2\pi k/6$, and the s_k's are reflections.

(a) Find a subgroup of order 4 in D_6, if it exists.

(b) Find a non-cyclic subgroup of order 6 in D_6, if it exists.

4. Let G be a group and H, K be subgroups of G such that $K < H < G$. Suppose that $[G : H]$ and $[H : K]$ are finite. Show that $[G : K]$ is finite and we have

$$[G : K] = [G : H][H : K].$$

5. Prove that a group with at least 2 elements but containing no proper nontrivial subgroups must be cyclic and of prime order.