

THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH 2070A Algebraic Structures 2019-20
Homework 2
Due Date: 19th September 2019

Compulsory Part

1. Let G be a group. Show that for all $a, b \in G$ such that $|ab|$ is finite, we have $|ab| = |ba|$.
2. Let $\omega = e^{\pi i/12} \in \mathbb{C}$. Find the order of ω^8 , ω^{13} , ω^{22} and ω^{2019} respectively as elements in the multiplicative group \mathbb{C}^\times of the complex number.
3. Consider the permutations

$$\sigma = (1264)(2513), \tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 2 & 7 & 5 & 1 & 3 & 6 \end{pmatrix} \in S_7.$$

- (a) Express σ and τ as
 - i. a product of transpositions, and
 - ii. a product of disjoint cycles.
 - (b) Compute $|\sigma|$, $|\tau|$ and $|\sigma\tau|$. (*Hint*: You may use the result in Q3(c) in the Optional Part.)
4. How many elements are of order 2 in D_6 ?

Optional Part

1. Let a, b be elements of a group G . Suppose a has order 5 and $a^3b = ba^3$. Prove that $ab = ba$.
2. By definition, the **orthogonal group** $O(2, \mathbb{R})$ consists of real 2×2 matrices A which satisfy the condition

$$A^t A = A A^t = I,$$

where A^t denotes the transpose of A , and I is the 2×2 identity matrix.

- (a) Show that $O(2, \mathbb{R})$ is a group under matrix multiplication.
 - (b) Find an element of order 2 in $O(2, \mathbb{R})$.
 - (c) Find an element of order 3 in $O(2, \mathbb{R})$.
3. (a) Show that a k -cycle in S_n has order k
- (b) Let $\mu_1, \mu_2 \in S_n$ be two disjoint cycles. Using the fact that disjoint cycles commute, show that

$$|\mu_1\mu_2| = \text{lcm}(|\mu_1|, |\mu_2|).$$

(c) By induction, show that if $\mu_1, \mu_2, \dots, \mu_r \in S_n$ are disjoint cycles, then

$$|\mu_1 \mu_2 \cdots \mu_r| = \text{lcm}(k_1, k_2, \dots, k_r),$$

where $k_i = |\mu_i|$ for $i = 1, 2, \dots, r$.

4. Find the order of the following elements in S_7 :

(a) (1325).

(b) (1325)(47).

(c) (1325)(647).

(d) (35)(46)(37)(32).

5. (a) How many elements are of order 3 in S_5 ?

(b) How many elements are of order 4 in S_6 ?

6. Consider the dihedral group D_n , where n is an integer greater than 2. For $j = 0, 1, 2, \dots, n-1$, let r_j denote the anticlockwise rotation about the origin by $2\pi j/n$.

(a) Show that for any $k \in \mathbb{N}$, rotation $r \in D_n$ and reflection $s \in D_n$, we have

$$sr^k s = (sr s)^k.$$

(b) Show that for any reflection $s \in D_n$ and any rotation $r \in D_n$, we have

$$sr s = r^{-1}.$$

(Hint: First prove the identities for one particular reflection s , then use the fact that any other reflection s' is equal to sr' for some rotation $r' \in D_n$.)

7. Let G be a group in which every element has order at most 2. Show that G is abelian.

8. Let G be a finite group with an even number of elements. Show that there must be an order 2 element $a \in G$.