1 Compact Sets in $\mathbb{R}$

Throughout this section, let $(x_n)$ be a sequence in $\mathbb{R}$. Recall that a subsequence $(x_{n_k})_{k=1}^{\infty}$ of $(x_n)$ means that $(n_k)_{k=1}^{\infty}$ is a sequence of positive integers satisfying $n_1 < n_2 < \cdots < n_k < n_{k+1} < \cdots$, that is, such sequence $(n_k)$ can be viewed as a strictly increasing function $n: k \in \{1, 2, \ldots\} \mapsto n_k \in \{1, 2, \ldots\}$.

In this case, note that for each positive integer $N$, there is $K \in \mathbb{N}$ such that $n_K \geq N$ and thus we have $n_k \geq N$ for all $k \geq K$.

Let us first recall the following two important theorems in real line.

**Theorem 1.1 Nested Intervals Theorem** Let $(I_n := [a_n, b_n])$ be a sequence of closed and bounded intervals. Suppose that it satisfies the following conditions.

(i) $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$.

(ii) $\lim_{n \to \infty} (b_n - a_n) = 0$.

Then there is a unique real number $\xi$ such that $\bigcap_{n=1}^{\infty} I_n = \{\xi\}$.

*Proof:* See [1, Theorem 2.5.2, Theorem 2.5.3]. \hfill $\square$

**Theorem 1.2 (Bolzano-Weierstrass Theorem)** Every bounded sequence in $\mathbb{R}$ has a convergent subsequence.

*Proof:* See [1, Theorem 3.4.8]. \hfill $\square$

**Definition 1.3** A subset $A$ of $\mathbb{R}$ is said to be compact (more precise, sequentially compact) if every sequence in $A$ has a convergent subsequence with the limit in $A$.

We are now going to characterize the compact subsets of $\mathbb{R}$. The following is an important notation in mathematics.

**Definition 1.4** A subset $A$ is said to be closed in $\mathbb{R}$ if it satisfies the condition:

if $(x_n)$ is a sequence in $A$ and $\lim x_n$ exists, then $\lim x_n \in A$.

**Example 1.5** (i) $\{a\}; [a, b]; [0, 1] \cup \{2\}; \mathbb{N}$; the empty set $\emptyset$ and $\mathbb{R}$ all are closed subsets of $\mathbb{R}$. 

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(ii) \((a, b)\) and \(\mathbb{Q}\) are not closed.

The following Proposition is one of the basic properties of a closed subset which can be directly shown by the definition. So, the proof is omitted here.

**Proposition 1.6** Let \(A\) be a subset of \(\mathbb{R}\). The following statements are equivalent.

(i) \(A\) is closed.

(ii) For each element \(x \in \mathbb{R} \setminus A\), there is \(\delta_x > 0\) such that \((x - \delta_x, x + \delta_x) \cap A = \emptyset\).

The following is an important characterization of a compact set in \(\mathbb{R}\). **Warning:** this result is not true for the so-called metric spaces in general.

**Theorem 1.7** Let \(A\) be a closed subset of \(\mathbb{R}\). Then the following statements are equivalent.

(i) \(A\) is compact.

(ii) \(A\) is closed and bounded.

**Proof:** It is clear that the result follows if \(A = \emptyset\). So, we assume that \(A\) is non-empty.

For showing \((i) \Rightarrow (ii)\), assume that \(A\) is compact.

We first claim that \(A\) is closed. Let \((x_n)\) be a sequence in \(A\). Then by the compactness of \(A\), there is a convergent subsequence \((x_{n_k})\) of \((x_n)\) with \(\lim_k x_{n_k} \in A\). So, if \((x_n)\) is convergent, then \(\lim_n x_n = \lim_k x_{n_k} \in A\). Therefore, \(A\) is closed.

Next, we are going to show the boundedness of \(A\). Suppose that \(A\) is not bounded. Fix an element \(x_1 \in A\). Since \(A\) is not bounded, we can find an element \(x_2 \in A\) such that \(|x_2 - x_1| > 1\).

Similarly, there is an element \(x_3 \in A\) such that \(|x_3 - x_k| > 1\) for \(k = 1, 2\). To repeat the same step, we can obtain a sequence \((x_n)\) in \(A\) such that \(|x_n - x_m| > 1\) for \(m \neq n\). From this, we see that the sequence \((x_n)\) does not have a convergent subsequence. In fact, if \((x_n)\) has a convergent subsequence \((x_{n_k})\). Put \(L := \lim_k x_{n_k}\). Then we can find a pair of sufficient large positive integers \(p\) and \(q\) with \(p \neq q\) such that \(|x_{n_p} - L| < 1/2\) and \(|x_{n_q} - L| < 1/2\). This implies that \(|x_{n_p} - x_{n_q}| < 1\). It leads to a contradiction because \(|x_{n_p} - x_{n_q}| > 1\) by the choice of the sequence \((x_n)\). Thus, \(A\) is bounded.

It remains to show \((ii) \Rightarrow (i)\). Suppose that \(A\) is closed and bounded.

Let \((x_n)\) be a sequence in \(A\). Thus, \((x_n)\). Then the Bolzano-Weierstrass Theorem assures that there is a convergent subsequence \((x_{n_k})\). Then by the closeness of \(A\), \(\lim_k x_{n_k} \in A\). Thus \(A\) is compact.

The proof is finished.

\(\square\)

For convenience, we call a collection of open intervals \(\{J_\alpha : \alpha \in \Lambda\}\) an **open intervals cover** of a given subset \(A\) of \(\mathbb{R}\), where \(\Lambda\) is an arbitrary non-empty index set, if each \(J_\alpha\) is an open interval (not necessary bounded) and

\[ A \subseteq \bigcup_{\alpha \in \Lambda} J_\alpha. \]
Theorem 1.8 Heine-Borel Theorem: Any closed and bounded interval \([a, b]\) satisfies the following condition:

\[(HB)\] Given any open intervals cover \(\{J_{\alpha}\}_{\alpha \in \Lambda}\) of \([a, b]\), we can find finitely many \(J_{\alpha_1}, \ldots, J_{\alpha_N}\) such that \([a, b] \subseteq J_{\alpha_1} \cup \cdots \cup J_{\alpha_N}\).

Proof: Suppose that \([a, b]\) does not satisfy the above Condition \((HB)\). Then there is an open intervals cover \(\{J_{\alpha}\}_{\alpha \in \Lambda}\) of \([a, b]\) but it has no finite sub-cover. Let \(I_1 := [a_1, b_1] = [a, b]\) and \(m_1\) the mid-point of \([a_1, b_1]\). Then by the assumption, \([a_1, m_1]\) or \([m_1, b_1]\) cannot be covered by finitely many \(J_{\alpha}\)’s. We may assume that \([a_1, m_1]\) cannot be covered by finitely many \(J_{\alpha}\)’s.

Put \(I_2 := [a_2, b_2] = [a_1, m_1]\). To repeat the same steps, we can obtain a sequence of closed and bounded intervals \(I_n = [a_n, b_n]\) with the following properties:

(a) \(I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots\);
(b) \(\lim_n (b_n - a_n) = 0\);
(c) each \(I_n\) cannot be covered by finitely many \(J_{\alpha}\)’s.

Then by the Nested Intervals Theorem, there is an element \(\xi \in \bigcap_n I_n\) such that \(\lim_n a_n = \lim_n b_n = \xi\). In particular, we have \(a = a_1 \leq \xi \leq b_1 = b\). So, there is \(\alpha_0 \in \Lambda\) such that \(\xi \in J_{\alpha_0}\). Since \(J_{\alpha_0}\) is open, there is \(\varepsilon > 0\) such that \((\xi - \varepsilon, \xi + \varepsilon) \subseteq J_{\alpha_0}\). On the other hand, there is \(N \in \mathbb{N}\) such that \(a_N\) and \(b_N\) in \((\xi - \varepsilon, \xi + \varepsilon)\) because \(\lim_n a_n = \lim_n b_n = \xi\). Thus we have \(I_N = [a_N, b_N] \subseteq (\xi - \varepsilon, \xi + \varepsilon) \subseteq J_{\alpha_0}\). It contradicts to the Property (c) above. The proof is finished.

\[\square\]

Remark 1.9 The assumption of the closeness and boundedness of an interval in Heine-Borel Theorem is essential.

For example, notice that \(\{J_n := (1/n, 1) : n = 1, 2, \ldots\}\) is an open interval covers of \((0, 1)\) but you cannot find finitely many \(J_n\)’s to cover the open interval \((0, 1)\).

The following is a very important feature of a compact set.

Theorem 1.10 Let \(A\) be a subset of \(\mathbb{R}\). Then the following statements are equivalent.

(i) \textbf{Heine-Borel property:} For any open intervals cover \(\{J_{\alpha}\}_{\alpha \in \Lambda}\) of \(A\), we can find finitely many \(J_{\alpha_1}, \ldots, J_{\alpha_N}\) such that \(A \subseteq J_{\alpha_1} \cup \cdots \cup J_{\alpha_N}\).

(ii) \(A\) is compact.

(iii) \(A\) is closed and bounded.

Proof: The result will be shown by the following path

\[(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (i)\]

For \((i) \Rightarrow (ii)\), assume that the condition \((i)\) holds but \(A\) is not compact. Then there is a sequence \((x_n)\) in \(A\) such that \((x_n)\) has no subsequent which has the limit in \(A\). Put \(X =\)
\{x_n : n = 1, 2, \ldots \}. Then \(X\) is infinite. Also, for each element \(a \in A\), there is \(\delta_a > 0\) such that \(J_a := (a - \delta_a, a + \delta_a) \cap X\) is finite. Indeed, if there is an element \(a \in A\) such that \((a - \delta, a + \delta) \cap A\) is infinite for all \(\delta > 0\), then \((x_n)\) has a convergent subsequence with the limit \(a\). On the other hand, we have \(A \subseteq \bigcup_{a \in A} J_a\). Then by the compactness of \(A\), we can find finitely many \(a_1, \ldots, a_N\) such that \(A \subseteq J_{a_1} \cup \cdots \cup J_{a_N}\). Then by the choice of \(J_a\)'s, \(X\) must be finite. This leads to a contradiction. Therefore, \(A\) must be compact.

The implication \((ii) \Rightarrow (iii)\) follows from Theorem 1.7 at once.

It remains to show \((iii) \Rightarrow (i)\). Suppose that \(A\) is closed and bounded. Then we can find a closed and bounded interval \([a, b]\) such that \(A \subseteq [a, b]\). Notice that for each element \(x \in [a, b] \setminus A\), there is \(\delta_x > 0\) such that \((x - \delta_x, x + \delta_x) \cap A = \emptyset\) since \(A\) is closed by using Proposition 2.4. If we put \(I_x = (x - \delta_x, x + \delta_x)\) for \(x \in [a, b] \setminus A\), then we have

\[
[a, b] \subseteq \bigcup_{a \in A} J_a \cup \bigcup_{x \in [a, b] \setminus A} I_x.
\]

Using the Heine-Borel Theorem 1.8, we can find finitely many \(J_a\)'s and \(I_x\)'s, say \(J_{a_1}, \ldots, J_{a_N}\) and \(I_{x_1}, \ldots, I_{x_K}\), such that \(A \subseteq [a, b] \subseteq J_{a_1} \cup \cdots \cup J_{a_N} \cup I_{x_1} \cup \cdots \cup I_{x_K}\). Note that \(I_x \cap A = \emptyset\) for each \(x \in [a, b] \setminus A\) by the choice of \(I_x\). Therefore, we have \(A \subseteq J_{a_1} \cup \cdots \cup J_{a_N}\) and hence \(A\) is compact.

The proof is finished. \(\square\)

**Remark 1.11** In fact, the condition in Theorem 1.10(i) is the usual definition of a **compact set** for a general topological space. More precise, if a set \(A\) satisfies the Definition 1.4, then \(A\) is said to be **sequentially compact**. Theorem 1.10 tells us that the notation of the compactness and the sequentially compactness are the same as in the case of a subset of \(\mathbb{R}\). However, these two notation are different for a general topological space.

**Strongly recommended:** take the courses: MATH 3060; MATH3070 for the next step.

## 2 Appendix: Open subsets of \(\mathbb{R}\)

**Definition 2.1** Let \(V\) be a subset of \(\mathbb{R}\).

(i) A point \(c \in V\) is called an interior point of \(V\) if there is \(r > 0\) such that \((c - r, c + r) \subseteq V\).

(ii) \(V\) is said to be an open subset of \(\mathbb{R}\) is for every element in \(V\) is an interior point of \(V\).

In this case, if \(x_0 \in V\), then \(V\) is called an open neighborhood of the point \(x_0\).

**Example 2.2** With the notation as above, we have

(i) All open intervals are open subsets of \(\mathbb{R}\).

(ii) \(\emptyset\) and \(\mathbb{R}\) are open subsets.

(iii) Any closed and bounded interval is not an open subset.

(iv) The set of all rational numbers \(\mathbb{Q}\) is neither open nor closed subset.
Proposition 2.3 A non-empty subset $A$ of $\mathbb{R}$ is open if and only if there is sequence of open intervals $I_n = (a_n, b_n)$ for $n = 1, 2, \ldots$ such that $A = \bigcup_{n=1}^{\infty} I_n$ and $I_n \cap I_m = \emptyset$ for $m \neq n$.

Proof: Assume that $A$ is an open subset. Notice that $\overline{\mathbb{Q}} = \mathbb{R}$. Since $A$ is open, we see that $A \cap \mathbb{Q}$ is also a non-empty countable subset. Let $A \cap \mathbb{Q} = \{x_1, x_2, \ldots\}$. For each $x_k$, put $I_k := \bigcup \{J : x_k \in J \text{ and } J \text{ is an open interval}\}$. Then $X = \bigcup_{k=1}^{\infty} I_k$. On the other hand, we notice that $I_k$ is also any open interval (Why??). From this, we see that $I_k \cap I_j = \emptyset$ if $k \neq j$. Thus, we can find a subsequence $(x_{n_k})$ such that $I_{n_k} \cap I_{n_j} = \emptyset$ for $k \neq j$. Thus the sequence of disjoint open intervals $(I_{n_k})_{k=1}^{\infty}$ that we want. The converse is clear. \qed

Proposition 2.4 Let $A$ be a subset of $\mathbb{R}$. Then the following statements are equivalent.

(i) $A$ is closed.

(ii) If $(x_n)$ is a sequence in $A$ and $\lim x_n$ exists, then $\lim x_n \in A$.

The following an important relation between the notion of openness and closeness.

Proposition 2.5 A subset $A$ of $\mathbb{R}$ is open if and only if its complement $A^c = \mathbb{R} \setminus A$ is closed in $\mathbb{R}$.

Proof: For $(\Rightarrow)$, we assume that $A$ is open first. If $A^c$ does not have the limit points, then the set $A^c$ is clearly a closed set by the definition. Now let $c$ be a limit point of $A^c$. \qed

References