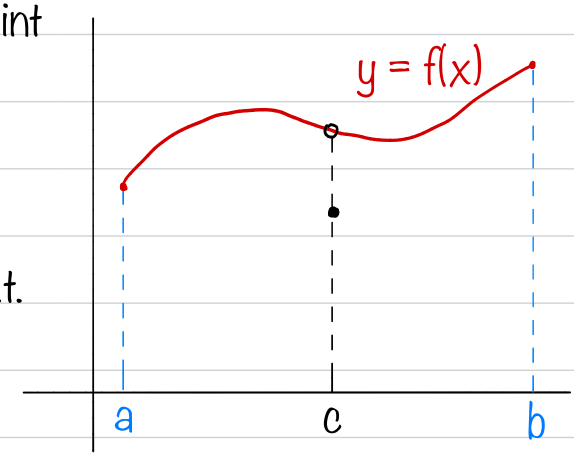
MATH2060B TUTORIAL 6

More examples on integrable functions:

Example 1: Let $f \in \mathcal{R}[a, b]$. Show that if we modify the function at finitely many points, the new function is still integrable on $[a, b]$. Moreover, it has the same integral as f.

Solution: Note that it suffices to show that only one point of f is modified. (Why?) In the case, let $\tilde{f}$ be the modified function with modified value at c. i.e.,

$$\tilde{f}(x) = f(x) \qquad \forall\, x \in [a, b] \setminus \{c\}.$$



Let $\varepsilon > 0$. We need to construct a partition P of $[a, b]$ s.t.

$$\sum \omega_i(\tilde{f}, P)\Delta x_i^{(P)} < \varepsilon$$

Consider the 3 cases separately:

Case 1: $a = c$. (Exercise) Case 2: $b = c$. (Exercise)

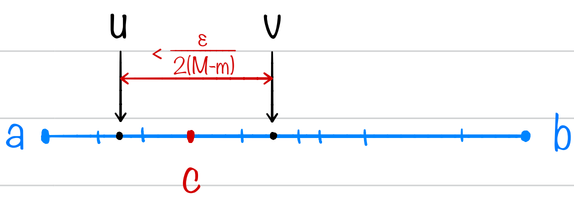
Case 3: $a < c < b$. Since f is integrable, there exists a partition Q of $[a, b]$ such that

$$\sum \omega_i(f, Q)\Delta x_i^{(Q)} < \varepsilon/2$$

(Note that Δx_i here correspond to the partition Q.)

Fix u, v such that $a < u < c < v < b$ and

$$v - u < \frac{\varepsilon}{2(M-m)}$$



Consider the partition $P = Q \cup \{u, v\}$, then the sum $\sum \omega_i(\tilde{f}, P)\Delta x_i^{(P)}$ can be divided into:

* $\sum$ over intervals $[x_{i-1}, x_i] \subseteq [u, v]$: $\quad \Sigma_1 \leqslant \sum (M-m)\Delta x_i^{(P)} = (M-m)\sum \Delta x_i^{(P)}$
* $\sum$ over remaining intervals: $\quad \Sigma_2 \leqslant \sum \omega_i(f, Q)\Delta x_i^{(Q)}$

It follows that

$$\sum \omega_i(\tilde{f}, P)\Delta x_i^{(P)} = \Sigma_1 + \Sigma_2 \leqslant (M-m)(v-u) + \sum \omega_i(f, Q)\Delta x_i^{(Q)}$$

$$< (M-m)\frac{\varepsilon}{2(M-m)} + \frac{\varepsilon}{2} = \varepsilon$$

The same idea can be used to show that $\tilde{f}$ and f have the same integral by considering the upper sum and lower sum. I leave it as an exercise. #

Remark: This shows that the integral reflects the global property of functions.

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7.1.5 Theorem *Suppose that f and g are in $\mathcal{R}[a, b]$. Then:*

(a) If $k \in \mathbb{R}$, *the function kf is in $\mathcal{R}[a, b]$ and*

$$\int_a^b kf = k \int_a^b f.$$

(b) *The function $f + g$ is in $\mathcal{R}[a, b]$ and*

$$\int_a^b (f + g) = \int_a^b f + \int_a^b g.$$

(c) *If $f(x) \leq g(x)$ for all $x \in [a, b]$, then*

$$\int_a^b f \leq \int_a^b g.$$

Remark: (a) and (b) gives the vector space structure of $\mathcal{R}[a, b]$.
(c) gives the order preserving property of the integral. In particular,

$$\left| \int_a^b f \right| \leq \int_a^b |f| \qquad \text{(triangle inequality for integrals)}$$

7.2.9 Additivity Theorem *Let $f := [a, b] \to \mathbb{R}$ and let $c \in (a, b)$.* Then $f \in \mathcal{R}[a, b]$ *if and only if its restrictions to $[a, c]$ and $[c, b]$ are both Riemann integrable. In this case*

$$\int_a^b f = \int_a^c f + \int_c^b f. \tag{6}$$

Remark: It guarantees that if f is integrable on any subinterval of [a, b].

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Fundamental Theorem of Calculus:

7.3.1 Fundamental Theorem of Calculus (First Form) *Suppose there is a* ***finite*** *set* $\mathbf{E}$ *in* $[a, b]$ *and functions* $f, F := [a, b] \to \mathbb{R}$ *such that:*

(a) F *is continuous on* $[a, b]$,

(b) $F'(x) = f(x)$ *for all* $x \in [a, b] \backslash E$,

(c) f *belongs to* $\mathcal{R}[a, b]$.

F : anti-derivative of f

Then we have

$$\int_a^b f = F(b) - F(a). \tag{1}$$

7.3.3 Definition If $f \in \mathcal{R}[a, b]$, then the function defined by

$$F(z) := \int_a^z f \qquad \text{for} \qquad z \in [a, b], \tag{3}$$

is called the **indefinite integral** of f with **basepoint** a. (Sometimes a point other than a is used as a basepoint; see Exercise 6.)

7.3.4 Theorem *The indefinite integral* F *defined by* (3) *is continuous on* $[a, b]$. *In fact, if* $|f(x)| \le M$ *for all* $x \in [a, b]$, *then* $|F(z) - F(w)| \le M|z - w|$ *for all* $z, w \in [a, b]$.

7.3.5 Fundamental Theorem of Calculus (Second Form) *Let* $f \in \mathcal{R}[a, b]$ *and let* f *be continuous at a point* $c \in [a, b]$. *Then the indefinite integral, defined by* (3), *is differentiable at* c *and* $F'(c) = f(c)$.

7.3.6 Theorem *If* f *is continuous on* $[a, b]$, *then the indefinite integral* F, *defined by* (3), *is differentiable on* $[a, b]$ *and* $F'(x) = f(x)$ for all $x \in [a, b]$.

Remark: The indefinite integral of f may not be an anti-derivative of f.

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Illustrations:

FTC I

$C[a, b] \quad\quad R[a, b]$

$\cup \quad\quad \cup$

Assumption $F \longmapsto F' = f$

Conclusion $\int_a^b F' = F(b) - F(a)$

$$\boxed{\int_a^x \frac{d}{dt} f(t)\, dt = f(x) - f(a)}$$

FTC 2

$C[a, b] \quad\quad R[a, b]$

$\cup \quad\quad \cup$

Assumption $F = \int_a^x f \longleftarrow f$

Conclusion F is cts. If f is cts, then $F' = f$.

$$\boxed{\frac{d}{dx} \int_a^x f(t)\, dt = f(x)}$$

Example 2: Evaluate the integral $\int_0^\pi \cos x\, dx$

Solution: Let $f(x) = \cos x$. Note that if we take $F(x) = \sin x$, then $F'(x) = f(x)$.
Then by FTC,

$$\int_0^\pi \cos x\, dx = \int_0^\pi f = F(\pi) - F(0) = (\sin \pi) - (\sin 0) = 0$$

#

Remark: This shows that if we know an anti-derivative of a given function. It is easy to calculate its integral.

Example 3: Show that any anti-derivative of f must be differ by a constant.

Solution: We need to show that if F_1 and F_2 are both anti-derivatives of f, i.e.,

$$F_1' = f = F_2'$$

then there exists some constant c such that $F_1(x) - F_2(x) = c \quad \forall\, x \in [a, b]$.

Notice that $(F_1 - F_2)' = F_1' - F_2' = f - f = 0$. The result follows. #

Remark: This shows that the choice of the base point in the formula of anti-derivative is not important.

Exercises:

1. Show there does not exist a continuously differentiable function f on [0, 2] such that $f(0) = -1$, $f(2) = 4$, and $f'(x) \leq 2$ for $0 \leq x \leq 2$. (Apply the Fundamental Theorem.)

Solution: It is obvious that we need to prove the assertion by contradiction.
Suppose such function f on [0, 2] exists.

(Method 1) Using MVT:

By MVT, there exists some $c \in (0, 2)$ such that

$$f(2) - f(0) = f'(c)(2 - 0)$$

It follows that

$$2 \geqslant f'(c) = \frac{f(2) - f(0)}{2 - 0} = \frac{4 - (-1)}{2 - 0} = 2.5$$

This is a contradiction.

(Method 2) Using FTC:

By FTC, we have

$$\int_0^2 f'(x)\,dx = f(2) - f(0) = 5$$

On the other hand, since $f'(x) \leqslant 2$ for all x [0, 2], then

$$\int_0^2 f'(x)\,dx \leqslant \int_0^2 2 = 4$$

This is a contradiction.