# MATH2060B TUTORIAL 4

## L'Hospital's Rule:

**6.3.2 Cauchy Mean Value Theorem** *Let $f$ and $g$ be continuous on $[a, b]$ and differentiable on $(a, b)$, and assume that $g'(x) \neq 0$ for all $x$ in $(a, b)$. Then there exists $c$ in $(a, b)$ such that*

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}.$$

Remark: The proof of the Cauchy Mean Value Theorem is similar to that of the Mean Value Theorem, which are consequences of the Rolle's Theorem.

**6.3.3 L'Hospital's Rule, I** *Let $-\infty \leq a < b \leq \infty$ and let $f$, $g$ be differentiable on $(a, b)$ such that $g'(x) \neq 0$ for all $x \in (a,b)$. Suppose that*

(1) $$\lim_{x\to a+} f(x) = 0 = \lim_{x\to a+} g(x).$$

**(a)** *If* $\displaystyle\lim_{x\to a+} \frac{f'(x)}{g'(x)} = L \in \mathbb{R}$, *then* $\displaystyle\lim_{x\to a+} \frac{f(x)}{g(x)} = L.$

**(b)** *If* $\displaystyle\lim_{x\to a+} \frac{f'(x)}{g'(x)} = L \in \{-\infty, \infty\}$, *then* $\displaystyle\lim_{x\to a+} \frac{f(x)}{g(x)} = L.$

Remark: The statement of the theorem seems complicated. Always keep in mind that:

* The limit of f'(x)/g'(x) must exist. If such limit DNE, we cannot use the theorem to conclude that the limit of f(x)/g(x) DNE.
* This part deals with the case 0/0. In fact, we have another part to deal with the case $\infty/\infty$.

**6.3.5 L'Hospital's Rule, II** *Let $-\infty \leq a < b \leq \infty$ and let $f$, $g$ be differentiable on $(a, b)$ such that $g'(x) \neq 0$ for all $x \in (a,b)$. Suppose that*

(5) $$\lim_{x\to a+} g(x) = \pm\infty.$$

**(a)** *If* $\displaystyle\lim_{x\to a+} \frac{f'(x)}{g'(x)} = L \in \mathbb{R}$, *then* $\displaystyle\lim_{x\to a+} \frac{f(x)}{g(x)} = L.$

**(b)** *If* $\displaystyle\lim_{x\to a+} \frac{f'(x)}{g'(x)} = L \in \{-\infty, \infty\}$, *then* $\displaystyle\lim_{x\to a+} \frac{f(x)}{g(x)} = L.$

**Example 1:** Evaluate $\lim_{x\to 0} \frac{\arctan x}{x}$.

**Solution:** To be extra careful about the conditions required. Let's check them all.

Let $f(x) = \arctan x$ and $g(x) = x$. Both $f$ and $g$ are differentiable on $(0, 1)$.

We first compute their derivatives:

$$f'(x) = \frac{1}{1+x^2} \qquad g'(x) = 1$$

Therefore $g'(x) \neq 0$ on $(0, 1)$. Also, we have

- $\lim_{x\to 0^+} f(x) = \arctan 0 = 0$
- $\lim_{x\to 0^+} g(x) = 0$

$$\left.\begin{array}{l}\lim_{x\to 0^+} f(x) = \arctan 0 = 0 \\ \lim_{x\to 0^+} g(x) = 0\end{array}\right\} \quad \boxed{\lim_{x\to 0^+} f(x) = 0 = \lim_{x\to 0^+} g(x)}$$

- $\lim_{x\to 0^+} \frac{f'(x)}{g'(x)} = \lim_{x\to 0^+} \frac{1}{1+x^2} = 1/(1+0) = 1$

Hence by L'Hospital's Rule 1(a), we have

$$\lim_{x\to 0^+} \frac{\arctan x}{x} = \lim_{x\to 0^+} f(x)/g(x) = \lim_{x\to 0^+} f'(x)/g'(x) = 1.$$

Similarly we can do the same thing on the interval $(-1, 0)$ and yields

$$\lim_{x\to 0^-} \frac{\arctan x}{x} = \lim_{x\to 0^-} f(x)/g(x) = \lim_{x\to 0^-} f'(x)/g'(x) = 1.$$

Thus the required limit is 1. #

**Remark:** Generally, we deal with two-sided limits but the L'Hospital's Rules concerns on one-sided limits. No worry, we can still apply the theorem because the conditions is still implied by the two-sided limit, we can combine the results on one-sided limits for both directions to give the desired result on two-sided limits.

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Example 2: Evaluate $\lim_{x\to 0+} x^x$.

Solution: Note that for any $x > 0$, $x^x = e^{x\ln x}$.

Let $y = x^x$.
Then $\ln y = x\ln x$
$y = e^{x\ln x}$

We need to calculate the limit of $x \ln x$:

$$\lim_{x\to 0+} x\ln x = \lim_{x\to 0+} \underbrace{\frac{\ln x}{1/x}}_{\frac{\infty}{\infty}} \underset{\text{L'Hospital's II}}{=} \underbrace{\boxed{\lim_{x\to 0+} \frac{1/x}{-1/x^2}}}_{\text{make sure it exists first}} = -\lim_{x\to 0+} x = 0.$$

It follows that the required limit is $e^0 = 1$. #

Theorem: Let $f$ be a real-valued function defined on $(a, b)$ and $c$ be a point in its domain.

(a) If $f'(c)$ exists, then

$$f'(c) = \lim_{h\to 0} \frac{f(c+h) - f(c-h)}{2h}$$

(b) If $f''(c)$ exists and $f$ is differentiable on $(a, b)$, then

$$f''(c) = \lim_{h\to 0} \frac{f(c+h) + f(c-h) - 2f(c)}{h^2}$$

Proof: (a) We cannot apply L'Hospital's Rule here because $f$ may not be differentiable near $c$.
Note that

$$\frac{f(c+h) - f(c-h)}{2h} = \frac{f(c+h) - f(c)}{2h} + \frac{f(c) - f(c-h)}{2h}$$

$$= \frac{f(c+h) - f(c)}{2h} - \frac{f(c) - f(c+k)}{2k} \quad \text{where } k = -h.$$

It follows that

$$\lim_{h\to 0} \frac{f(c+h) - f(c-h)}{2h} = \frac{1}{2}\lim_{h\to 0} \frac{f(c+h) - f(c)}{h} + \frac{1}{2}\lim_{k\to 0} \frac{f(c+k) - f(c)}{k} = f'(c)$$

(b) Since now that $f$ is differentiable on $(a, b)$, we can apply L'Hospital's Rule.

$$\lim_{h\to 0} \frac{f(c+h) + f(c-h) - 2f(c)}{h^2} = \lim_{h\to 0} \frac{f'(c+h) + f'(c-h)}{2h} = f''(c)$$

$$\left.\begin{array}{l} * \ F(h) = f(c+h) - f(c-h) - 2f(c) \\ * \ G(h) = h^2 \end{array}\right\} \text{functions of } h$$

# Taylor's Theorem:

**6.4.1 Taylor's Theorem** *Let $n \in \mathbb{N}$, let $I := [a, b]$, and let $f : I \to \mathbb{R}$ be such that $f$ and its derivatives $f', f'', \ldots, f^{(n)}$ are continuous on $I$ and that $f^{(n+1)}$ exists on $(a, b)$. If $x_0 \in I$, then for any $x$ in $I$ there exists a point $c$ between $x$ and $x_0$ such that*

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}. \tag{2}$$

Example 1: Show that $x - x^3/6 \leq \sin x \leq x + x^3/6 \quad \forall\, x > 0$.

Solution: Let $f(x) = \sin x$ and compute its derivatives.

$$f^{(n)}(x) = \begin{cases} \sin x, & \text{if } n = 4k \\ \cos x, & \text{if } n = 4k+1 \\ -\sin x, & \text{if } n = 4k+2 \\ -\cos x, & \text{if } n = 4k+3 \end{cases}$$

Thus $f^{(n)}(0) = 0, 1, 0, -1, 0, 1, \ldots$ for $n = 0, 1, 2, 3, 4, 5, \ldots$

Let $x > 0$ and fix some natural number $n$. By Taylor's Theorem for $x_0 = 0$, there exists $c \in (0, x)$ such that

$$f(x) = f(0) + f'(0)(x - 0) + \ldots + \frac{f^{(n)}(0)}{n!}(x - 0)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(x - 0)^{n+1} \tag{\#}$$

In particular if $n = 2$,

$$\sin x = 0 + 1x + \frac{0}{2!}x^2 + \frac{-\cos c}{3!}x^3 = x - \frac{\cos c}{6}x^3 \tag{$*$}$$

Note that since $-1 \leq \cos c \leq 1$ and $x > 0$, together with $(*)$, we have

$$-\frac{1}{6}x^3 \leq \sin x - x \leq \frac{1}{6}x^3$$

Thus implies $x - x^3/6 \leq \sin x \leq x + x^3/6$. #

Remark: Compare it with the inequality for sine in the previous tutorial, we get this stronger inequality.

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Example 2: Estimate sin(0.5) using the above inequalities.

Solution: If we use the weaker one, we know that $-0.5 \leq \sin(0.5) \leq 0.5$.

If we use the stronger one, we know that

$$0.5 - (0.5)^3/6 \leq \sin(0.5) \leq 0.5 + (0.5)^3/6$$

By computations,

$$0.5 - (0.5)^3/6 = 23/48 = 0.479... \geq 0.47$$

$$0.5 + (0.5)^3/6 = 25/48 = 0.520... \leq 0.53$$

Hence $0.47 \leq \sin(0.5) \leq 0.53$. This approximation is much better. #

Remark: The error in this estimation is bounded by $(0.5)^3/6 = 1/48 = 0.020...$

Example 3: Estimate sin(0.5) with error less than $10^{-5}$.

Solution: Obviously, the previous inequality is not applicable because the error is still too big.

We need to apply Taylor's Theorem with a larger n. How big should it be?

Notice that by (#) in Example 1,

$$f(x) - P_n(x) = R_n(x),$$

where $P_n$ is the n-th Taylor polynomial we uses to approximate f and $R_n$ is the remainder term which controls the error!

Hope: $< \text{error} = 10^{-5}$

In this example, we have

$$|\sin(0.5) - P_n(0.5)| = |R_n(0.5)| = \frac{|f^{(n+1)}(c)|}{(n+1)!}(0.5)^{n+1} \leq \boxed{\frac{1}{2^{n+1}(n+1)!}}$$

Thus we required $2^{n+1}(n+1)! > 100000$.

Note that $\underbrace{2^6 \times 6!}_{n=5} = 46080$ and $\underbrace{2^7 \times 7!}_{n=6} = 645120$, we choose $n = 6$.

It follows that sin(0.5) can be approximated by

$$P_6(0.5) = 0.5 - (0.5)^3/6 + (0.5)^5/5!$$

$$= 1/2 - 1/48 + 1/3840$$

$$= 0.479427..... \qquad \#$$

Remark: sin(0.5) = 0.479425538, first 5 digits are correct!

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