Differentiability

Differentiation is one of the main focus in MATH2060. Let's start with introducing the notion of derivative.

Definition (c.f. Definition 6.1.1)**.** Let $I \subseteq \mathbb{R}$ be an interval containing c and $f : I \to \mathbb{R}$ be a function. A real number L is said to be the **derivative** of f at c if for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\left| \frac{f(x) - f(c)}{x - c} - L \right| < \varepsilon, \quad \text{whenever } x \in I \text{ and } 0 < |x - c| < \delta.$$

In this case, we say that f is **differentiable** at c and denote $L = f'(c)$.

Remark. By comparing to the definition of limit of functions, f is differentiable at c with derivative L if and only if the following limits exist and equal L:

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \lim_{h \to 0} \frac{f(c + h) - f(c)}{h}$$

One of the famous and foundation result of differentiable functions is that differentiability implies continuity. Hence a function fails to be continuous at a point implies that it fails to be differentiable at that point.

Theorem (c.f. Theorem 6.1.2)**.** *If $f : I \to \mathbb{R}$ has a derivative at $c \in I$, f is continuous at c.*

Exercise 1. Show that the following functions are differentiable at $x = 1$ and find their derivatives:

(a) $f(x) = x^2$ (b) $f(x) = \sqrt{x}$ (c) $f(x) = \sin x$

Solution. We try to use different approaches to show the results.

(a) Let's use ε-δ notation to show that the derivative of x^2 at 1 is 2. Note that

$$\left| \frac{x^2 - 1^2}{x - 1} - 2 \right| = \left| \frac{x^2 - 1 - 2x + 2}{x - 1} \right| = \left| \frac{(x - 1)^2}{x - 1} \right| = |x - 1|, \quad \text{whenever } x \neq 1.$$

Let $\varepsilon > 0$ and take $\delta = \varepsilon$. Then whenever $0 < |x - 1| < \delta$,

$$\left| \frac{x^2 - 1^2}{x - 1} - 2 \right| = |x - 1| < \delta = \varepsilon.$$

Hence $f'(1) = 2$.

(b) Let's use limit notation to show that the derivative of $\sqrt{x}$ at 1 is $\frac{1}{2}$. Note that

$$\frac{\sqrt{x} - \sqrt{1}}{x - 1} = \frac{\sqrt{x} - 1}{x - 1} \cdot \frac{\sqrt{x} + 1}{\sqrt{x} + 1} = \frac{1}{\sqrt{x} + 1}, \quad \text{for } x \neq 1.$$

Hence we have

$$\lim_{x \to 1} \frac{\sqrt{x} - \sqrt{1}}{x - 1} = \lim_{x \to 1} \frac{1}{\sqrt{x} + 1} = \frac{1}{\sqrt{1} + 1} = \frac{1}{2}.$$

It follows that $\sqrt{x}$ is differentiable at 1 with derivative $\frac{1}{2}$.

(c) Again we use limit notation to find the derivative of $\sin x$. Note that

$$\lim_{h\to 0}\frac{\sin(1+h)-\sin 1}{h}=\lim_{h\to 0}\frac{2}{h}\cos\left(\frac{2+h}{2}\right)\sin\left(\frac{h}{2}\right)=\cos\left(\frac{2+0}{2}\right)\cdot\lim_{h\to 0}\frac{\sin(h/2)}{h/2}.$$

Using the fact that $(\sin h)/h \to 1$ as $h\to 0$, we can conclude that $\sin x$ is differentiable at 1 with derivative $\cos 1$.

Exercise 2. Show that the following functions are not differentiable at 0:

(a) $f(x)=|\sin x|$, (b) $g(x)=\sqrt[3]{x}$.

Solution. We can observe the graphs of the functions and see that they are not differentiable. (Try to draw them!)

(a) We need to show that the following limit does not exists:

$$\lim_{x\to 0}\frac{|\sin x|-|\sin 0|}{x-0}=\lim_{x\to 0}\frac{|\sin x|}{x}$$

Observe that its left-hand and right-hand limit are not equal:

$$\lim_{x\to 0^-}\frac{|\sin x|}{x}=-\lim_{x\to 0^-}\frac{\sin x}{x}=-1 \quad\text{and}\quad \lim_{x\to 0^+}\frac{|\sin x|}{x}=\lim_{x\to 0^+}\frac{\sin x}{x}=1$$

Hence its limit does not exist.

(b) We need to show that the following limit does not exists:

$$\lim_{x\to 0}\frac{\sqrt[3]{x}}{x}=\lim_{x\to 0}\frac{1}{x^{2/3}}$$

Obviously, the function is unbounded near 0, the result follows.

Reading Exercise. Theorem 6.1.3. The technique to prove the product rule and quotient rule is important, please be familiar with it.

Chain Rule

There is a useful theorem to characterize the differentiability of a function a certain point. This theorem is essential to prove the chain rule.

Carathéodory's Theorem (c.f. 6.1.5). *Let f be defined on an interval I containing the point c. Then f is differentiable at c if and only if there exists a function φ on I that is continuous at c and*

$$f(x)-f(c)=\varphi(x)(x-c),\quad \forall x\in I.$$

In this case, we have $\varphi(c)=f'(c)$.

Reading Exercise. Theorem 6.1.5 and Theorem 6.1.6. The proof of the chain rule via Carathéodory's Theorem is also important.

Exercise 3 (c.f. Section 6.1, Ex.12)**.** If $r > 0$ is a rational number, let $f : [0, \infty) \to \mathbb{R}$ be defined by $f(x) = x^r \sin(1/x)$ for $x \neq 0$ and $f(0) = 0$. Determine those values of r for which $f'(0)$ exists.

Remark. In the textbook, the domain of f is $\mathbb{R}$, but x^r is undefined for $x < 0$.

Solution. We show that $f'(0)$ exists if and only if $r > 1$. First note that the difference quotient at 0 is given by

$$\frac{f(x) - f(0)}{x - 0} = \frac{x^r \sin(1/x) - 0}{x - 0} = x^{r-1} \sin(1/x), \quad \text{where } x > 0.$$

Hence

$$-x^{r-1} \leq \left| \frac{f(x) - f(0)}{x - 0} \right| \leq x^{r-1}, \quad \text{for } x \neq 0.$$

If $r > 1$, we can apply squeeze theorem and show that $f'(0) = \lim_{x \to 0} \dfrac{f(x) - f(0)}{x - 0} = 0$.

If $0 < r \leq 1$, then

$$\frac{f(x) - f(0)}{x - 0} = \frac{\sin(1/x)}{x^{1-r}}, \quad \text{for } x \neq 0.$$

We need to show that its limit does not exist as $x \to 0$. To see this, we consider the cases $r = 1$ and $0 < r < 1$ separately:

- If $r = 1$, consider the sequences (x_n) and (u_n) defined by

  $$x_n = \frac{1}{2\pi n + \pi/2} \quad \text{and} \quad u_n = \frac{1}{2\pi n}.$$

  Then

  $$\frac{f(x_n) - f(0)}{x_n - 0} = 1 \quad \text{and} \quad \frac{f(u_n) - f(0)}{u_n - 0} = 0.$$

  So the limit does not exists.

- If $0 < r < 1$, then using the sequnce (u_n) defined as above, we see that

  $$\frac{f(u_n) - f(0)}{u_n - 0} = \left(2\pi n + \frac{\pi}{2}\right)^{1-r},$$

  which is unbounded. Hence the limit does not exists.

Exercise 4 (c.f. Section 6.1, Ex.13)**.** If $f : \mathbb{R} \to \mathbb{R}$ is differentiable at $c \in \mathbb{R}$, show that

$$f'(c) = \lim_{n \to \infty} n(f(c + 1/n) - f(c)).$$

However, show by example that the existence of the limit of this sequence does not imply the existence of $f'(c)$.

Solution. If f is differentiable at c, then the following limit exists:

$$\lim_{h \to 0} \frac{f(c + h) - f(c)}{h}$$

In particular, by the sequential criterion of limit of function, take (x_n) to be $(1/n)$, the result follows. For a counter-example, consider the function $f(x) = |x|$ and $c = 0$.