

THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH2050C Mathematical Analysis I
Tutorial 9 (April 8)

Squeeze Theorem. Let $A \subseteq \mathbb{R}$, let $f, g, h: A \rightarrow \mathbb{R}$, and let c be a cluster point of A . If

$$f(x) \leq g(x) \leq h(x) \quad \text{for all } x \in A, x \neq c,$$

and if $\lim_{x \rightarrow c} f = L = \lim_{x \rightarrow c} h$, then $\lim_{x \rightarrow c} g = L$.

Example 1. Deduce the following limits using the Squeeze Theorem.

(a) $\lim_{x \rightarrow 0} \cos x = 1$

(b) $\lim_{x \rightarrow 0} \left(\frac{\cos x - 1}{x} \right) = 0$

(c) $\lim_{x \rightarrow 0} \left(\frac{\sin x}{x} \right) = 1$

(d) $\lim_{x \rightarrow 0} (x \sin(1/x)) = 0$.

Definition. Let $A \subseteq \mathbb{R}$, let $f: A \rightarrow \mathbb{R}$, and let $c \in A$. We say that f is **continuous at c** if, given any $\varepsilon > 0$, there exists $\delta > 0$ such that if $x \in A$ and $|x - c| < \delta$, then $|f(x) - f(c)| < \varepsilon$.

Remarks. (1) If $c \in A$ is a cluster point of A , then f is continuous at c if and only if

$$f(c) = \lim_{x \rightarrow c} f(x).$$

(2) If $c \in A$ is not a cluster point of A , then f is automatically continuous at c since $A \cap V_\delta(c) = \{c\}$ for some $\delta > 0$.

Sequential Criterion for Continuity. A function $f: A \rightarrow \mathbb{R}$ is continuous at the point $c \in A$ if and only if for every sequence (x_n) in A that converges to c , the sequence $(f(x_n))$ converges to $f(c)$.

Example 2. Let $A := \mathbb{R}$ and let f be the Dirichlet's function defined by

$$f(x) := \begin{cases} 1 & \text{if } x \text{ is rational,} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

Show that f is not continuous at any point of \mathbb{R} .

Classwork

1. Determine the limit $\lim_{x \rightarrow 0} \frac{1}{x} \sin(\sin(x^2))$.
2. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be defined by

$$f(x) := \begin{cases} x & \text{if } x \text{ is rational,} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

Determine all the continuity point(s) of f . Justify your answer.