THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH2050C Mathematical Analysis I

Tutorial 2 (January 22)

The following were discussed in the tutorial this week:

1 The Completeness Property of $\mathbb{R}$

Definition. Let S be a nonempty subset of $\mathbb{R}$. Suppose S is bounded above. Then $u \in \mathbb{R}$ is said to be a **supremum** of S if it satisfies the conditions:

(i) u is an upper bound of S (that is, $s \leq u$ for all $s \in S$), and

(ii) if v is any upper bound of S, then $u \leq v$.

Here (ii) is equivalent to

(ii)' if $v < u$, then there exists $s_v \in S$ such that $v < s_v$.

Remark. (1) u may or may not be an element of S.

(2) The number u is unique and we write $\sup S = u$.

(3) $\inf S$ can be defined similarly provided S is bounded below.

Example 1. Let $A := \{x \in \mathbb{R} : 1/x > x\}$. Find $\sup A$ and $\inf A$, if they exist.

Solution. Note that

$$x \in A \iff \frac{x^2-1}{x} < 0 \iff \frac{(x-1)(x+1)}{x} < 0 \iff x < -1 \text{ or } 0 < x < 1.$$

Thus $A = (-\infty, -1) \cup (0, 1)$.

It is easy to see that A is not bounded below, so $\inf A$ does not exist.

Next we want to show that $\sup A = 1$. Clearly

$$x < 1 \quad \text{for all } x \in A.$$

So 1 is an upper bound of A. Let $v < 1$.

Want: v is not an upper bound of A, that is $\exists\, s_v \in A$ s.t. $s_v > v$.

Take $s_v := \max\{(v+1)/2, 1/2\}$. Then

$$0 < 1/2 \leq s_v < 1,$$

so that $s_v \in A$. Moreover,

$$s_v \geq (v+1)/2 > (v+v)/2 = v.$$

Hence $\sup A = 1$. ◀

The Completeness Property of $\mathbb{R}$. *Every nonempty set of real numbers that has an upper bound also has a supremum in $\mathbb{R}$.*

Example 2. (a) Let S be a nonempty subset of $\mathbb{R}$ that is bounded above, and let a be any real number in $\mathbb{R}$. Define the set $a + S := \{a + s : s \in S\}$. Show that

$$\sup(a + S) = a + \sup S.$$

(b) Let A and B be nonempty subsets of $\mathbb{R}$ that satisfy the property:

$$a \leq b \qquad \text{for all } a \in A \text{ and } b \in B.$$

Show that $\sup A \leq \inf B$.

Example 3. Suppose that f and g are real-valued functions with common domain $D \subseteq \mathbb{R}$. We assume that f and g are bounded (that is, $f(D)$ and $g(D)$ are bounded).

(a) If $f(x) \leq g(x)$ for all $x \in D$, show that $\sup f(D) \leq \sup g(D)$.

(b) If $f(x) \leq g(y)$ for all $x, y \in D$, show that $\sup f(D) \leq \inf g(D)$.

Classwork

1. Fill in the blanks to complete the following definition of an infimum.

   *Suppose $S \subseteq \mathbb{R}$ is nonempty and ____________. Then $w \in \mathbb{R}$ is said to be an **infimum** of S if it satisfies the conditions:*

   (i) ____________________________ for all $s \in S$, and

   (ii) if $w < v$, then there exists ____.

2. Let $A := \{x \in \mathbb{R} : |x + 2| + |1 - x| > 5\} \cap \{x \in \mathbb{R} : x \geq 0\}$.

   (a) What are the elements of the set A?

   (b) Is A bounded above? Is A bounded below?

   (c) Find $\sup A$ and $\inf A$, if they exist. Justify your answer.

3. Let S be a nonempty subset of $\mathbb{R}$ that is bounded above. Prove that

$$\inf\{-s : s \in S\} = -\sup S.$$

Solution. Let $-S := \{-s : s \in S\}$. For any $s \in S$, we have $s \leq \sup S$, so that $-\sup S \leq -s$. Hence $-\sup S$ is a lower bound of $-S$.

If v is any lower bound of $-S$, then $v \leq -s$ for any $s \in S$. Then $s \leq -v$ for any $s \in S$, and hence $-v$ is an upper bound of S. By the definition of supremum, we have $\sup S \leq -v$, so that $v \leq -\sup S$. Therefore $-\sup S$ is the greatest lower bound of $-S$, that is $\inf(-S) = -\sup S$. ◀